More on Arithmetic Functions

MME 529 Worksheet for April 25, 2017

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This note adds some more research questions for students to present in the May 2 class.

We ended our April 25 meeting with the following definition.

An arithmetic function $\alpha : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if it satisfies

$$\alpha(mn) = \alpha(m) \cdot \alpha(n)$$

whenever m and n are relatively prime. I.e., we only require this equation when m and n have no common factor: $\gcd(m, n) = 1$. [We might get lucky and have properties like $\alpha(8) = \alpha(2) \cdot \alpha(4)$, but we do not care if such things are true and we don't care if they are false: our function is multiplicative as long as this nice relation holds <u>at least</u> when m and n are relatively prime.]

10. For each function α in our original table, define a new function β by the rule

$$\beta(n) = \sum_{d|n} \alpha(d).$$

That is, $\beta(n)$ is obtained by adding up all the values of α over the positive divisors of n. For example, $\beta(6) = \alpha(1) + \alpha(2) + \alpha(3) + \alpha(6)$. Do any of these functions look familiar to you? Do you see any nice properties?

11. For each function α in our original table, define a new function γ by the rule

$$\gamma(n) = \sum_{d|n} \alpha(d)\mu(\frac{n}{d})$$

where μ is the Möbius function. For example, $\beta(6) = \alpha(1)\mu(6) + \alpha(2)\mu(3) + \alpha(3)\mu(2) + \alpha(6)\mu(1)$. Do any of *these* functions look familiar to you? Again, check for nice properties.

12. What can you say about the *Liouville function* $\lambda(n)$ defined by

$$\lambda(n) = (-1)^m$$

where n has prime factorization $n = q_1 q_2 \ldots q_m$ where each q_j is prime. For example, if we look at $n = 18$, we have $n = 2 \cdot 3 \cdot 3$ so $m = 3$ (total number of prime factors, counting multiplicity) and we have $\lambda(18) = (-1)^3 = -1$.

13. Based on the multiplicative property and what you can discover about prime powers, derive expressions for $\nu(n)$, $\sigma(n)$, $\phi(n)$ and $\mu(n)$ in terms of the prime power decomposition of n, which we may agree to write as

$$n = p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}.$$