Group Isomorphisms

MME 529 Worksheet for May 23, 2017
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Goal: *Illustrate the power of abstraction by seeing how groups arising in different contexts are really the same.*

There are many different kinds of groups, arising in a dizzying variety of contexts. Even on this worksheet, there are too many groups for any one of us to absorb. But, with different teams exploring different examples, we should – as a class – discover some justification for the study of groups in the abstract.

The Integers Modulo n: With John Goulet, you explored the additive structure of $\mathbb{Z}_n$. Write down the addition table for $\mathbb{Z}_5$ and $\mathbb{Z}_6$. These groups are called *cyclic* groups: they are generated by a single element, the element 1, in this case. That means that every element can be found by adding 1 to itself an appropriate number of times.

The Group of Units Modulo n: Now when we look at $\mathbb{Z}_n$ using multiplication as our operation, we no longer have a group. (Why not?) The group

$$U(n) = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$$

is sometimes written $\mathbb{Z}_n^*$ and is called the *group of units* modulo n. An element in a number system (or ring) is a "unit" if it has a multiplicative inverse. Write down the multiplication tables for $U(6)$, $U(7)$, $U(8)$ and $U(12)$.

The Group of Rotations of a Regular n-Gon: Imagine a regular polygon with n sides centered at the origin O. Let e denote the identity transformation, which leaves the polygon entirely fixed and let a denote a rotation about O in the counterclockwise direction by exactly $360/n$ degrees ($2\pi/n$ radians). Then, applying this twice, we find that a^2 is a counterclockwise rotation by $720/n$ degrees. Find the *order* of a — the smallest positive integer k such that $a^k = e$ — and write down the Cayley table for the group $Z_n = \{e, a, a^2, \ldots, a^{k-1}\}$.

The Dihedral Group D_n: A regular polygon with n sides has $2n$ symmetries, including n rotations and n reflections. Write down the Cayley tables for D_4 and D_5.

The Group $GL(2, \mathbb{F})$ of Invertible 2×2 Matrices: For $\mathbb{F} = \mathbb{Z}_2$, work out the Cayley table for the group of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries from $\mathbb{F}$ satisfying $ad - bc \neq 0$. The next case is $\mathbb{F} = \mathbb{Z}_3$, but that Cayley table is too large to work out by hand since it has $8 \cdot 6 = 48$ elements. Discuss the case where $\mathbb{F}$ is the field of real numbers or rational numbers.

The External Direct Product: If G and H are groups, then the Cartesian product $G \times H$ becomes a group by applying the two operations coordinatewise. Gallian denotes this group by $G \oplus H$. For example, $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$: $(1, 1) + (1, 1) = (0, 2)$,

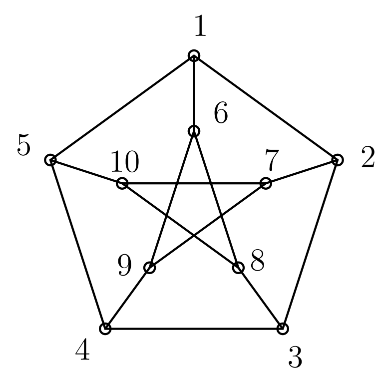
$(1,1) + (1,1) + (1,1) = (1,0)$, etc. Work out the addition tables for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Diagrammatically describe the Cayley table for $\mathbb{Z}_2 \oplus S_3$.

The Symmetry Group of a Graph: The graph P below is called the *Petersen graph.* It has ten *vertices* and 15 *edges*; two vertices joined by an edge are said to be *adjacent.* An *automorphism* of P is a permutation of the vertices of P which preserves adjacency. That is,

$$\varphi : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 10\}$$

bijectively (i.e., the function is both one-to-one and onto) in such a way that vertex i is adjacent to vertex j if and only if $\varphi(i)$ is adjacent to $\varphi(j)$. How many symmetries does the Petersen graph have?

P



The Quaternion Group: The field of complex numbers $\mathbb{C}$ consists of all expressions of the form $a + bi$ where a and b are real numbers and i is a symbol which satisfies only the rule $i^2 = -1$. The nonzero complex numbers form an abelian group under multiplication and the complex numbers of modulus one — i.e., those $z = a + bi$ with $|z|^2 = a^2 + b^2 = 1$ — form what is called the "circle group". Consider the subgroup of the circle group consisting of ± 1 and $\pm i$. Construct a Cayley table (multiplication table, in this case) for this group of order four. What familiar group is it isomorphic to?

Now consider the division ring of *quaternions*

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

where now we imagine size distinct square roots of -1. We define symbols i, j and k satisfying

$$i^2 = j^2 = k^2 = -1, \qquad ij = k, \quad ji = -k, \qquad jk = i, \quad kj = -i, \qquad ki = j, \quad ik = -j.$$

Work out the multiplication table for the "quaternion group"

$$Q_8 = \{1,\ -1,\ i,\ -i,\ j,\ -j,\ k,\ -k\}.$$

The Sporadic Finite Simple Groups: In addition to cyclic groups of prime order, there are seventeen infinite families of finite simple groups[1] and exactly 26 exceptions. These 26

[1]A group is *simple* if it does not "break up" further into smaller groups – more precisely, G is simple if every nontrivial group homomorphism $G \to H$ is one-to-one.

“sporadic finite simple groups” were discovered between 1861 and 1976 or so. Use MAPLE’s ”GroupTheory” package to find the order (size) of as many of these as you can. A full list can be found in Wikipedia.