Arithmetic Functions

MME 529 Worksheet for April 25, 2017
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Goal: *Discover patterns and structure in some arithmetic functions considered in class.*

The set of *natural numbers* is

$$\mathbb{N} = \{1,\ 2,\ 3,\ 4,\ \ldots\},$$

i.e., the natural numbers are the positive integers. These numbers have been studied for millenia and yet we still discover new things about them every year.

An *arithmetic function* is a function defined on the set of natural numbers. That is, a function that takes the form

$$\alpha : \mathbb{N} \to \mathbb{C}$$

where $\mathbb{C}$ denotes the *complex numbers*. Of course there are boatloads of arithmetic functions: we can map a positive integer n to $\sqrt{n}$, the number of positive multiples of thirteen that are less than n and do not end in a "1", and so on. But which ones are important? History gives us some guidance here.

In this worksheet, we compute the following functions:

- the *number of divisors* function $\nu(n)$ which counts the number of natural numbers d satisfying $d|n$
- the *sum of divisors* function $\sigma(n)$ which is defined as the sum of all natural numbers d which divide n
- *Euler's totient function* (or the *Euler phi function*) is defined as

  $$\phi(n) = |\{a\ :\ 1 \le a \le n,\ \gcd(a, n) = 1\}|\,.$$

  This is the number of integers in the set $\{1, 2, \ldots, n\}$ which have no common factor with n (except 1, of course)

The goal of this worksheet is to explore these functions, to discover an important property that they share, and to discover other arithmetic functions which also enjoy this special property.

Let's compute some values: each team work on filling in one part of this table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\nu(n)$	1											6	2	4						
$\sigma(n)$	1											28	14	24						
$\phi(n)$	1											4	12	6						

Now we have enough data to start discovering theorems!

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\nu(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28	14	24	24	31	18	39	20	42
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1	1	1	0	-1	0	-1	0

Questions:

1. What is $\alpha(1)$ for each of these functions α (here, α can be ν, σ, etc.)
2. What is $\alpha(p)$ when p is prime?
3. What happens to prime powers? Can you predict $\alpha(p^k)$ when p is prime and k is a natural number?
4. I've added another row into the table. This is the Möbius function $\mu(n)$. Can you find the pattern and reverse-engineer a definition for $\mu(n)$?
5. For every function α that we have listed (including the top row! …where $\alpha(n) = n$), we notice that $\alpha(12) = \alpha(3) \cdot \alpha(4)$ with $12 = 3 \cdot 4$, but $\alpha(16)$ is not always equal to $\alpha(4)$ times $\alpha(4)$. Do you see any other triples of columns where this rule applies?

Do not proceed to this next part until we all resolve the last question.

Let's finish by discovering other multiplicative arithmetic functions. (I've left room in the table for your discoveries!)

1. Which constant functions $\alpha(n) = b$ are multiplicative?
2. Which functions of the form $\alpha(n) = mn$ are multiplicative (where m is any complex number)?
3. Which linear functions $mx + b$ — or $\alpha(n) = mn + b$ for m and b any complex numbers — are multiplicative?
4. What is the sum of $\mu(d)$ over all positive divisors d of a natural number n? Do you see a pattern?