MA502 Linear Algebra
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MA502 Assignment 6

DUE DATE: Tuesday November 28, 2017, at the beginning of class.

Please carefully read and observe the presentation rules for assignments in this course.

1.) Prove that every Hermitian matrix M can be expressed as the difference $M = A - B$ of two positive semidefinite matrices (i.e., $A, B \succeq O$). Is it always possible to choose $A, B \succ O$?

2.) Let M be an $n \times n$ complex Hermitian matrix and let $W \subseteq \mathbb{C}^n$ be a τ_M-invariant subspace. Prove that the Rayleigh quotient $R_M(\mathbf{v}) = \frac{\langle \mathbf{v}, M\mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$ (where $\mathbf{v} \neq \mathbf{0}$) takes both its maximum and minimum values on W at eigenvectors of M. Show that the max and min are eigenvalues of M.

3.) Let M be an $n \times n$ complex Hermitian matrix and let $W \subseteq \mathbb{C}^n$ be a j-dimensional subspace. If the eigenvalues of M are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, prove that W contains a non-zero vector $\mathbf{v}$ with $R_M(\mathbf{v}) \leq \lambda_j$. *[HINT: Consider the subspace U spanned by eigenvectors $\mathbf{v}_j, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$ where $M\mathbf{v}_\ell = \lambda_\ell \mathbf{v}_\ell$.]*

4.) Let M be an $n \times n$ real symmetric matrix and suppose we are trying to find the eigenvectors and eigenvalues of M. Given an "approximate" eigenvector $\mathbf{v}$, we consider the best possible approximation to the corresponding eigenvalue to be the real number λ which minimizes the norm $\|M\mathbf{v} - \lambda\mathbf{v}\|$. By instead minimizing the squared norm $\|M\mathbf{v} - \lambda\mathbf{v}\|^2 = \langle M\mathbf{v} - \lambda\mathbf{v}, M\mathbf{v} - \lambda\mathbf{v} \rangle$, derive an expression for λ using elementary calculus. *[NOTE: Both M and $\mathbf{v}$ are constants here, so we have a quadratic function of one real variable λ and the minimum is obtained by setting the derivative equal to zero.]*

5.) Let $\tau : V \to V$ be a normal operator on the finite-dimensional inner product space V over $\mathbb{C}$ and let τ^* denote the adjoint operator. Prove: if $\lambda \in \mathbb{C}$ is an eigenvalue of τ with corresponding eigenspace V_λ, then $\bar{\lambda}$ is an eigenvalue of τ^* and the corresponding eigenspace is V_λ. *[HINT: For a complex number λ, what is another way to say that λ is not an eigenvalue of τ?]*