Solution to Homework 4

MA502 Linear Algebra

Problem 1.
Use elementary row operation to compute the determinants of following matrices.

(a) $A = \begin{bmatrix} i & i & i \\ 1 & -1 & 0 \\ -i & 1 & 0 \end{bmatrix}$

SOLUTION:

$$A \xrightarrow[(1)+(3)]{(1)\times i+(2)} \begin{bmatrix} i & i & i \\ 0 & -2 & -1 \\ 0 & 1+i & i \end{bmatrix} \xrightarrow{(2)\times\frac{1+i}{2}+(3)} \begin{bmatrix} i & i & i \\ 0 & -2 & -1 \\ 0 & 0 & -\frac{1-i}{2} \end{bmatrix} := A_1$$

Then, $det(A) = det(A_1) = i \cdot (-2) \cdot (-\frac{1-i}{2}) = 1 + i$.

(b) $B = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{1}{2} & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & \frac{1}{4} \\ 2 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$

SOLUTION:

$$\begin{aligned} B &\xrightarrow{(1)\leftrightarrow(5)} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & -\frac{1}{2} & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & \frac{1}{4} \\ 2 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -\frac{3}{2} & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -\frac{3}{4} \\ 0 & -3 & 1 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} \end{bmatrix} \\ &\xrightarrow[(4)\leftrightarrow(5)]{(3)\leftrightarrow(4)} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -\frac{3}{2} & 0 & 2 & -1 \\ 0 & -3 & 1 & 1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & -\frac{3}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -\frac{3}{2} & 0 & 2 & -1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & -\frac{3}{4} \end{bmatrix} := B_1 \end{aligned} \tag{1}$$

Hence, $det(B) = -det(B_1) = -\frac{9}{16}$.

(c) $C = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ in $\mathbb{F}_5$

SOLUTION:

$$C \xrightarrow{(2)-(1)\times 4} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} := C_1$$

Hence, $det(C) = det(C_1) = 0$

Problem 3.
Let $A = [a_{ij}]$ be an $n \times n$ upper-triangular matrix. Prove that the characteristic polynomial of A is

$$\chi_A(z) = (z - a_{11}) \cdots (z - a_{nn})$$

Proof. Let $\lambda = a_{nn}$. Since $A - \lambda I$ has a row of zeros, it has non-zero kernel. More specifically, there exists a vector $\mathbf{v}_n$ with $A\mathbf{v}_n = \lambda \mathbf{v}_n$ and the last entry of $\mathbf{v}_n$ is non-zero. Let $\hat{A}$ denote the matrix obtained by deleting the last row and column of A. The same argument gives us a vector $\mathbf{w}_{n-1}$ with nonzero $(n-1)^{\text{st}}$ entry such that $\hat{A}\mathbf{w}_{n-1} = a_{n-1,n-1}\mathbf{w}_{n-1}$. Extending this with one zero gives us a vector $\mathbf{v}_{n-1}$ with nonzero $(n-1)^{\text{st}}$ entry such that $A\mathbf{v}_{n-1} = a_{n-1,n-1}\mathbf{v}_{n-1}$. Not finished. □

Problem 4.
Problem 10.10 on Page 249 in Hobart's notes.

Proof. We first compute the characteristic polynomial by row reduction. We note that a row swaps just change the sign of the determinant and this can be ignored when computing the polynomial. So we proceed:

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 3/2 & 0 & 3/2 \\ -1 & 6-\lambda & 1 & 0 \\ 0 & 3/2 & 6-\lambda & 3/2 \\ 1 & 0 & -1 & 6-\lambda \end{bmatrix} \xrightarrow{(1)\leftrightarrow(4)} \begin{bmatrix} 1 & 0 & -1 & 6-\lambda \\ -1 & 6-\lambda & 1 & 0 \\ 0 & 3/2 & 6-\lambda & 3/2 \\ 6-\lambda & 3/2 & 0 & 3/2 \end{bmatrix}$$

$$\xrightarrow{(1)\leftrightarrow(4)} \begin{bmatrix} 1 & 0 & -1 & 6-\lambda \\ -1 & 6-\lambda & 1 & 0 \\ 0 & 3/2 & 6-\lambda & 3/2 \\ 6-\lambda & 3/2 & 0 & 3/2 \end{bmatrix}$$

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We can find by simple row reduction that the only eigenvalue is $\lambda = 6$. Hence,

$$A - \lambda I = \begin{bmatrix} 0 & 1.5 & 0 & 1.5 \\ -1 & 0 & 1 & 0 \\ 0 & 1.5 & 0 & 1.5 \\ 1 & 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (A - \lambda I)^2 = 0$$

We need to find independent vectors that is not in the kernel of $A - \lambda I$, and belongs to kernel of $(A-\lambda I)^2$. Such vectors can be as the following:

$$v_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T, v_2 = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$$

Hence, $v_{11} = (A-\lambda I)v_1 = \begin{bmatrix} 3 & 0 & 3 & 0 \end{bmatrix}^T$, $v_{22} = (A-\lambda I)v_2 = \begin{bmatrix} 0 & -2 & 0 & 2 \end{bmatrix}^T$. Hence, we can have $S = \begin{bmatrix} v_{11} & v_1 & v_{22} & v_2 \end{bmatrix}^{-1}$, and $J = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \end{bmatrix}$, such that $A = S^{-1}JS$. □

Problem 5. Problem 10.15 on Page 249 in Hobart's notes.

(a) *Proof.* For A, we have invertible matrix P, and Jordan matrix J such that $A = PJP^{-1}$. Then for an eigenvalue μ with order t, we have that for a generalized eigenvector v

$$(A-\mu I)^t v = (PJP^{-1} - \mu I)^t v = (P(J-\mu I)P^{-1})^t v = P(J-\mu I)^t P^{-1} v = 0$$
$$(A-\mu I)^{t-1} v = P(J-\mu I)^{t-1} P^{-1} v \neq 0$$

Meanwhile, since B is similar to A, we can find invertible matrix P_1 such that $B = P_1 A P_1^{-1}$. Then, $B = (P_1 P)J(P_1 P)^{-1}$, and choose $v_1 = P_1 v$.(v_1 cannot be 0, since otherwise, column space of P_1 is linearly dependent, contradicting with invertibility of P_1.) We will show that v_1 is a generalized eigenvector.

$$(B-\mu I)^t v_1 = P_1 P(J-\mu I)^t P^{-1} P_1^{-1} v_1 = P_1 P(J-\mu I)^t P^{-1} v = 0$$
$$(B-\mu I)^{t-1} v_1 = P_1 P(J-\mu I)^{t-1} P^{-1} v \neq 0$$

The second equation is due to the fact that P_1 is invertible, so column space is linear independent. Then we prove that for the same μ, A and B have the same order. Meanwhile, any generalized eigenvector of B with eigenvalue μ is linear combination of $\{P_1 v : v \text{ is generalized eigenvector of } A \text{ with } \mu\}$. If not, suppose v_2 is not in the span of $\{P_1 v\}$, and then we can find $P_1^{-1} v_2$, which is generalized eigenvector of A. Contradiction!

□

(b) *Proof.* The above statement is equivalent to "If for eigenvalue μ, A and B has different order of generalized eigenvector, then they are not similar matrices." Hence, we only need to prove that they have different order.

$$(C-5I)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } (C-5I)^3 = 0$$

$$D - 5I = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } (D-5I)^2 = 0$$

Hence, C has order 3, and D has order 2, and they are not similar matrices. $\square$