MA502 Linear Algebra
W. J. Martin
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MA502 Assignment 3

DUE DATE: Thursday, October 5, 2017, at the beginning of class.

Please carefully read and observe the presentation rules for assignments in this course.

1.) An $n \times n$ matrix A with entries from field $\mathbb{F}$ is *diagonalizable* over $\mathbb{F}$ if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(a) Prove that A is diagonalizable over field $\mathbb{F}$ if and only if $V = \mathbb{F}^n$ admits a basis each vector of which is an eigenvector for A.

(b) Using the fact that $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ whenever M is an $m \times n$ matrix and N is an $n \times m$ matrix, prove: *If A is diagonalizable, then $\mathrm{tr}\, A$ is the sum of all eigenvalues of A, counting multiplicities.*

2.) We compute the spectral decomposition of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

(a) Find and orthonormal basis for the eigenspace belonging to eigenvalue $\lambda = 1$.

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(c) Compute the orthogonal projection matrix E_λ from $\mathbb{R}^3$ onto the eigenspace V_λ for each eigenvalue λ of A and compute $\sum_\lambda \lambda E_\lambda$. Explain.

(d) Use part (b) to compute A^9.

3.) Consider the vector space $V = \mathcal{C}([0, \frac{\pi}{2}])$ of real-valued continuous functions on the interval $[0, \frac{\pi}{2}]$. Any bivariate continuous function $A(x, y)$ determines a linear operator $\tau : V \to V$ given by

$$\tau f(x) = \int_0^{\pi/2} A(x, y) f(y)\, dy \ .$$

In this exercise, we study $A(x, y) = \cos(x + y)$.

(a) Find all eigenfunctions $f(x)$ for τ that lie in the two-dimensional subspace

$$S = \{a \sin x + b \cos x \mid a, b \in \mathbb{R}\} \, .$$

[For this problem, use μ for the value $\frac{1}{4}(\pi^2 - 4)$.]

(b) Consider the subspace W of V spanned by $\{\cos(nx),\ \sin(nx)\}_{n=1}^{\infty}$. Is W a τ-invariant subspace? Explain. (You may need to look up some integration formulas from first-year calculus.)

4.) Consider the complex vector space V spanned by the following functions on the interval $[-\pi, \pi]$:

$$\mathbf{v}_1 = e^x + \cos x, \qquad \mathbf{v}_2 = e^x + \sin x, \qquad \mathbf{v}_3 = \sin x\ .$$

(I.e., V consists of all functions of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ where c_1, c_2, c_3 are complex numbers.) This space V is invariant under the differential operator $\tau \in \mathcal{L}(V)$ given by $\tau f = \frac{df}{dx}$.

(a) Compute the matrix $[\tau]_{\mathcal{B}}$ of τ with respect to basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Let us write $A = [\tau]_{\mathcal{B}}$. Using a computer, if necessary, find the eigenvalues of τ and determine a basis of eigenfunctions $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Write down the matrix $[\tau]_{\mathcal{C}}$.

(c) Can you choose an inner product on space V with respect to which operator τ is self-adjoint? Explain.

5(a) For which $n \times n$ matrices is the minimal polynomial linear? Explain.

(b) Find a 3×3 matrix with real entries which is not diagonalizable yet whose minimal polynomial is quadratic. *[HINT: You may assume the matrix is upper triangular. (Why?)]*

6.) Let S be a subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and let U be the $n \times k$ matrix with j^{th} column $\mathbf{v}_j$. In class, we showed that, when $\mathcal{B}$ is an orthonormal basis, $P = UU^\top$ is the matrix representing orthogonal projection of $\mathbb{R}^n$ onto S with respect to the standard basis.

(a) No longer assume that $\mathcal{B}$ is orthonormal, but that it is just any basis for S. Prove that the projection operator $\rho_{S,S^\perp}$ is represented by the matrix $P = U(U^\top U)^{-1}U^\top$.

(b) Does the matrix $P = UU^\top$ represent $\rho_{S,T}$ for some T? Explain.

(c) Now let V be the real vector space of polynomial functions $c_0 + c_1x + \cdots c_nx^n$ on the interval $[0, 1]$ with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ and let S be the space spanned by $\{1, x\}$. Find a simple expression for the orthogonal projection of V onto S with respect to this inner product.