# Solution to Homework 2

MA502 Linear Algebra

**Problem 1.**
Exercise 1 on P83 of Roman
Let $A \in \mathcal{M}_{m,n}$ have rank $k$. Prove that there are matrices $X \in \mathcal{M}_{m,k}$ and $y \in \mathcal{M}_{k,n}$, both of rank $k$, for which $A = XY$. Prove that $A$ has rank 1 if and only if it has form $A = x^t y$ where $x$ and $y$ are row matrices.

*Proof.* Suppose $A \in \mathcal{M}_{m,n}$ have rank $k$, and $A$ can be written as

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_k & v_{k+1} & \dots & v_n \end{bmatrix}$$

where $v_i$ is a $m \times 1$ vector. Meanwhile, we know that rank of its column space is also $k$. Hence, WOLOG, we can suppose $v_1 \dots v_k$ are linear independent, and $v_i (i > k)$ can be represented as a linear combination of $v_1 \dots v_k$. (Otherwise, we can multiply column exchange matrices to $A$, and they will not affect rank of a matrix.) Suppose for $j \geq k+1$,

$$v_j = \sum_{i=1}^{k} a_i^j v_i$$

Then define

$$V = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & \dots & 0 & a_1^{k+1} & \dots & a_i^n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & a_k^{k+1} & \dots & a_k^n \end{bmatrix}$$

Obviously, $rank(V) = rank(W) = k$. $V \in \mathcal{M}_{m,k}$, $W \in \mathcal{M}_{k,n}$, and $A = VW$.
For the second part, we need to prove $A$ has rank 1 iff $A = x^T y$, where, $x, y$ are row vectors.
Suppose $x, y$ are row vectors, and $A = x^T y$. Suppose $y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$. Then $A = \begin{bmatrix} y_1 x^T & y_2 x^T & \dots & y_n x^T \end{bmatrix}$, which obviously has rank 1.
Then, suppose $A$ has rank 1, and then $A$ can be represented as

$$A = \begin{bmatrix} k_1 x_1 & k_2 x_1 & \dots & k_n x_1 \end{bmatrix}$$

where $x_1$ is $m \times 1$ vector, and $k_i \in \mathbb{F}$. Then,

$$\begin{aligned} A &= x_1 \begin{bmatrix} k_1 & \dots & k_n \end{bmatrix} \\ &= x_1 y \\ &= (x_1^T)^T y \end{aligned} \tag{1}$$

where $x_1^T$, $y$ are row matrix, and $y = \begin{bmatrix} k_1 & \dots & k_n \end{bmatrix}$.
Hence, $A$ has rank 1 iff $A = x^T y$, where, $x, y$ are row vectors.

□

**Problem 2.**
Exercise 9 on P83 of Roman
Let $\tau \in \mathcal{L}(V)$ where $dim(V) < \infty$. If $rank(\tau^2) = rank(\tau)$ show that $Im(\tau) \cap ker(\tau) = \{0\}$

*Proof.* Since $Im(\tau^2) \subseteq Im(\tau)$, and $rank(\tau^2) = rank(\tau)$, we can get $Im(\tau^2) = Im(\tau)$. By Rank-Nullity Theorem, we can get $dim(ker(\tau^2)) = dim(ker(\tau))$. Meanwhile, $ker(\tau) \subseteq ker(\tau^2)$, and then $ker(\tau^2) = ker(\tau)$.
Suppose $x \in Im(\tau) \cap ker(\tau)$. Then, $\exists y \in V$ s.t. $x = \tau y$, and $\tau x = 0$. We can have $\tau^2 y = 0$, and $y \in ker(\tau^2)$. Since we have proven above that $ker(\tau^2) = ker(\tau)$, $y \in ker(\tau)$, and $\tau y = 0$, which means $x = 0$.
This proves that $\{0\} = Im(\tau) \cap ker(\tau)$.

□

**Problem 3.**
Exercise 13 on P84 of Roman
Let $\tau, \sigma \in \mathcal{L}$. Show that $rank(\tau + \sigma) \leq rank(\tau) + rank(\sigma)$.

*Proof.* Suppose $A$ is matrix of $\tau$, and $B$ is matrix of $\sigma$. Then, it is suffice to prove $rank(A + B) \leq rank(A) + rank(B)$. Denote $C_A$ as collection of column vectors of $A$.
Clearly, $span(C_{A+B}) \subseteq span(C_A \cup C_B)$, since $\forall w \in C_{A+B}, \exists v_1 \in C_A, v_2 \in C_B$ s.t $w = v_1 + v_2$.

$$\begin{aligned} rank(A + B) &\leq dim(C_A) + dim(C_B) - dim(C_A \cup C_B) \\ &\leq dim(C_A) + dim(C_B) \\ &= rank(A) + rank(B) \end{aligned} \tag{2}$$

□

**Problem 4.**
Consider the operator $\tau$ on $\mathcal{C}^\infty(\mathbb{R})$ defined by $\tau$: $f(x) \to f'(x)$ (first derivative). Describe all eigenvectors of operator $\tau$. Explain.

*Proof.* We claim that all eigenvectors (eigenfunctions) are in the following form: $f(x) = f(0)e^{\lambda x}$.
We know that $\tau f = f'$. We want to find eigenpairs by calculating $\tau f = \lambda f'$, where $\lambda$ is the eigenvalue. Hence, $f' = \lambda f \Rightarrow f(x) = f(0)e^{\lambda x}$, where $\lambda \in \mathbb{R}$.

□

**Problem 5.**

## a

Construct an example $A \in \mathcal{M}_{3\times 3}(\mathbb{C})$ with exactly one eigenvector up to equivalence under scalar multiplication. Choose $A$ so that the corresponding eigenvalue is $i$.

*Proof.* $A = \begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$. Eigenvalue is $i$, and eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. □

## b

Construct, or prove non-existence of $A \in \mathcal{M}_{3\times 3}(\mathbb{C})$ with no eigenvectors.

*Proof.* We claim that every square matrix with size $n$ in $\mathbb{C}$ has at least one eigenvector. Suppose that $A$ has size $n$, and choose $x \in \mathbb{C}^n$ that is not zero. Define

$$S = \{x, Ax, A^2x, \dots, A^n x\}$$

Since $S$ is a set of $n+1$ vectors in $\mathbb{C}^n$, it is linearly dependent. Suppose

$$a_0 x + a_1 Ax + a_2 A^2 x + \dots + a_n A^n x = 0$$

where $a_i \in \mathbb{C}$. By linearly dependence of vectors in $S$, there exists some $a_i \neq 0$. We need to prove that there are more than one $a_i \neq 0$. WOLOG, suppose $a_0 \neq 0$. If $a_1 = \dots = a_n = 0$. Then $a_0 x = 0$. We can get either $a_0 = 0$ or $x = 0$. Contradiction!
Let $m$ be the largest integer s.t. $a_m \neq 0$, where $m \geq 1$. WOLOG, we can suppose $a_m = 1$. Define

$$p(x) = a_0 + a_1 x + \dots + a_m x^m$$

Then $b_1, b_2, \dots, b_m \in \mathbb{C}$ s.t.

$$p(x) = (x - b_m)(x - b_{m-1}) \dots (x - b_1)$$

Then, we get

$$\begin{aligned} 0 &= a_0 x + a_1 Ax + \dots + a_n A^n x \\ &= a_0 x + a_1 Ax + \dots + a_m A^m x \\ &= (a_0 I_n + a_1 A + \dots + a_m A^m)x \\ &= p(A)x \\ &= (A - b_m I_n)(A - b_{m-1} I_n) \dots (A - b_1 I_n)x \end{aligned} \tag{3}$$

Let $k$ be the smallest integer s.t.

$$(A - b_k I_n)(A - b_{k-1} I_n) \dots (A - b_1 I_n)x = 0$$

We know that $k \leq m$. Define

$$z = (A - b_{k-1} I_n) \dots (A - b_1 I_n)x$$

Then, $z \neq 0$. If $k = 1$, then define $z = x$, which is still not zero.Then

$$(A - b_k I_n)z = (A - b_k I_n)(A - b_{k-1} I_n) \dots (A - b_1 I_n)x = 0$$

It gives that $Az = b_k z$. Since $z \neq 0$, it show that $A$ must have at least one eigenvector, as well as eigenvalue.

□

## c

In part (a), what are the other eigenvalues of your matrix $A$? Why do I know them already?

*Proof.* There is no other eigenvalue of matrix $A$. This is because different eigenvalues should have different eigenvectors, which will be proven in Question 8. Since there is only one eigenvector, then there should be only one eigenvalue.

□

**Problem 6.**
Write each of the following matrices as a linear combination of projection matrices:

## a

*Proof.*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

We can get its eigenvalues: 0 and 4. Consider $\lambda = 4$, and then eigenvector is $V = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Then its projection matrix is $P = V(V^T V)^{-1} V^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Hence, $A = 4P$. □

## b

*Proof.*

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

We can get its eigenvalues: $-2, 0, 2$. Consider $\lambda = -2$. Its eigenvector $V_{-2} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$. Then its projection matrix

$$P_{-2} = V_{-2}(V_{-2}^T V_{-2})^{-1} V_{-2}^T = \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Consider $\lambda = 2$. Its eigenvector $V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then its projection matrix

$$P_2 = V_2(V_2^T V_2)^{-1} V_2^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $B = (-2)P_{-2} + 2P_2$.

□

### c

*Proof.*

$$C = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 2 & 1 \end{bmatrix}$$

We can get its eigenvalues: $-2, 0, 8$. Consider $\lambda = -2$, and its eigenspace is $span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Consider

$$V_{-2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then, projection matrix

$$P_{-2} = V_{-2}(V_{-2}^T V_{-2})^{-1} V_{-2}^T = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Consider $\lambda = 8$. Its eigenvector $V_8 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then its projection matrix

$$P_8 = V_8(V_8^T V_8)^{-1} V_8^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, $C = (-2)P_{-2} + 8P_8$.

□

**Problem 7.**
Suppose $P$ and $Q$ are $n \times n$ projection matrices with real entries such that $PQ = 0$. What can you say about their column spaces? Explain.

*Proof.* Denote $C_A$ as collection of column vectors of $A$. Then column space of $A$ is $span(C_A)$. We claim that $span(C_P) \oplus span(C_Q) \subseteq \mathcal{M}_{n\times n}(\mathbb{R})$.This means:
1. $span(C_P) \cap span(C_Q) = \{0\}$
2. Union of two column spaces should be subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$.

First, let us prove $\mathcal{M}_{n\times n}(\mathbb{R}) = span(C_P) \oplus ker(P)$.
Suppose $x \in span(C_P) \cap ker(P)$. Then there exists $a$ such that $Px = 0$, and $x = Pa$, so $0 = Px = P^2a = Pa = x$. Hence, $span(C_W) \cap ker(W) = \{0\}$. Meanwhile, $dim(span(C_W)) + dim(ker(W)) = n$, and hence $\mathcal{M}_{n\times n}(\mathbb{R}) = span(C_P) \oplus ker(P)$.
It is obvious that $Px = 0$ iff $x \in ker(P)$. Since $PQ = 0$, every vector in $C_Q$ must be in $ker(P)$, and hence, $span(C_Q) \subseteq ker(P)$. Hence, $span(C_P) \oplus span(C_Q) \subseteq \mathcal{M}_{n\times n}(\mathbb{R})$.
Furthermore, if $PQ = QP = 0$ (which may not be true), by conversing $P$ and $Q$ in the above proof, we can get $span(C_P) \oplus span(C_Q) = \mathcal{M}_{n\times n}(\mathbb{R})$.

□

**Problem 8.**
Let $\tau$ be a linear operator on vector space $V$ over $\mathbb{F}$ and suppose $v_1, \dots, v_k$ are eigenvectors belonging to distinct eigenvalues: $\tau v_j = \lambda_j v_j$ with $\lambda_j \neq \lambda_l$ unless $j = l$. Prove that the set $S = \{v_1, \dots, v_k\}$ is linearly independent in $V$.

*Proof.* Suppose $A$ is matrix of $\tau$. Since $v_i \in N(A - \lambda_i I)$, it is suffice to prove for $\forall i$

$$N(A - \lambda_i I) \bigcap \sum_{j\neq i}^{k} N(A - \lambda_j I) = \{0\}$$

We can prove by induction. First, we can prove that it holds when $k = 2$. Suppose $x \in N(A - \lambda_1 I) \cap N(A - \lambda_2 I)$. Then,

$$
\begin{aligned}
&0 = (A - \lambda_1 I)x = (A - \lambda_2 I) \\
\Rightarrow &(\lambda_1 - \lambda_2)x = 0 \\
\Rightarrow &x = 0
\end{aligned} \tag{4}
$$

Last line is because $\lambda_1 \neq \lambda_2$. Hence, $N(A - \lambda_1 I) \cap N(A - \lambda_2 I) = \{0\}$.
Then, suppose that it hold when $k = n$. When $k = n + 1$. Suppose $x \in N(A - \lambda_{n+1} I) \cap \sum_{i=1}^{n} N(A - \lambda_j I)$. There exists $v_1, \ldots, v_n \in \bigcup_{j=1}^{n} N(A - \lambda_j I)$ such that $x = v_1 + \ldots + v_n$. WOLOG, suppose $v_i \in N(A - \lambda_i I)$. Then,

$$0 = (A - \lambda_{n+1} I)x = (A - \lambda_{n+1} I)(v_1 + \ldots + v_n)$$

Since $(A - \lambda_i I)v_i = 0$, for $i = 1, 2, \ldots, n$. We can have

$$(\lambda_{n+1} - \lambda_1)v_1 + \ldots + (\lambda_{n+1} - \lambda_n)v_n = 0$$

If there exists some $v_i \neq 0$, since $\{v_1, \ldots, v_n\}$ are linearly independent, there must exist some $\lambda_i = \lambda_{n+1}$. Contradiction! Hence, all $v_i$ is zero, and $x = 0$. The result holds when $k = n + 1$. Hence, $N(A - \lambda_i I) \bigcap \sum_{j \neq i}^{k} N(A - \lambda_j I) = \{0\}$, for $\forall i$. $S$ is linearly independent in $V$.

□