MA502 Linear Algebra
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MA502 Assignment 1

DUE DATE: Thursday, September 7, 2017, at the beginning of class.

Please carefully read the presentation rules below. **Any paper submitted which is sloppy or uses two sides of a page will be returned immediately with no credit.**

1.) Prove that the **sum** $\sum_{i\in\Lambda} S_i = \{\mathbf{v}_1 + \cdots + \mathbf{v}_n \mid \mathbf{v}_j \in \cup_{i\in\Lambda} S_i\}$ of any collection $\{S_i\}_{i\in\Lambda}$ of subspaces of a vector space V is a subspace. Show that $\sum_{i\in\Lambda} S_i$ is the least upper bound of the set $\{S_i\}_{i\in\Lambda}$ where subspaces are ordered under inclusion.

2.) Exercise 17 on p57 of Roman.

3.) Prove that every bijective linear transformation is a vector space isomorphism.

4.) Let $\mathbb{F}$ be a field in which $1 + 1 \neq 0$ and let $\mathcal{M}_n(\mathbb{F})$ denote the vector space of $n \times n$ matrices over $\mathbb{F}$. Prove that every matrix may be uniquely expressed as a sum of a symmetric matrix and a skew-symmetric matrix. Express this statement as a direct sum decomposition of the space.

5.) In each case **(a)**-**(c)**, a finite set $\mathcal{S}$ of vectors is given in a vector space. Let W denote the subspace spanned by $\mathcal{S}$. In each case, determine all subsets of $\mathcal{S}$ which form bases for W. (E.g., if $V = \mathbb{R}^3$ and

$$\mathcal{S} = \{\mathbf{v}_1 = (1,0,0), \mathbf{v}_2 = (0,1,0), \mathbf{v}_3 = (0,0,1), \mathbf{v}_4 = (1,1,0)\},$$

then $W = \text{span}(\mathcal{S}) = \mathbb{R}^3$ and the only bases contained in $\mathcal{S}$ are $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_3\}$, and $\{\mathbf{v}_4, \mathbf{v}_2, \mathbf{v}_3\}$.)

(a) $V = \mathcal{C}^\infty(\mathbb{R})$, $\quad \mathbf{v}_1 = \sin^2 t - 1$, $\mathbf{v}_2 = \cos^2 t + e^t$, $\mathbf{v}_3 = 4e^t - 1$, $\mathbf{v}_4 = 2e^t$.

(b) $V = \mathbb{Q}[t]$, $\quad \mathbf{v}_1 = t^2 - 3$, $\mathbf{v}_2 = t^2$, $\mathbf{v}_3 = 0$, $\mathbf{v}_4 = 1$.

(c) $V = \mathcal{M}_{2,2}(\mathbb{Z}_2)$, $\quad \mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

BASIC RULES FOR MA502 ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only one side of each sheet of paper;

III) Show your work. Explain your answers using FULL SENTENCES;

IV) Late assignments will, in general, not be accepted for credit.