Homework 1 Solution

September 14, 2017

1. Roman P42 Question 1

> $\star$ Prove that the **sum** $\sum_{i\in\Lambda} S_i = \{\mathbf{v}_1 + \cdots + \mathbf{v}_n \mid \mathbf{v}_j \in \cup_{i\in\Lambda} S_i\}$ of any collection $\{S_i\}_{i\in\Lambda}$ of subspaces of a vector space V is a subspace.

We know that $T := \sum_{i\in\Lambda} S_i$ consists of all finite sums of vectors in the union $\cup_{i\in\Lambda} S_i$ of the spaces. Looking at such a sum

$$\mathbf{v} = \mathbf{s}_1 + \cdots + \mathbf{s}_n,$$

it doesn't matter whether $\mathbf{s}_i$ and $\mathbf{s}_j$ come from the same – or different – subspaces in the collection. So if $\mathbf{w} = \mathbf{s}'_1 + \cdots + \mathbf{s}'_m$ we have

$$\mathbf{v} + \mathbf{w} = \mathbf{s}_1 + \cdots + \mathbf{s}_n + \mathbf{s}'_1 + \cdots + \mathbf{s}'_m$$

where each of the $n+m$ terms is a vector in $\cup_{i\in\Lambda} S_i$. By definition of the sum of subspaces, this, too, is a vector in T.

Next let $a \in \mathbb{F}$ be any scalar and let $\mathbf{v} = \mathbf{s}_1 + \cdots + \mathbf{s}_n$ be any vector in the sum of subspaces, since each $\mathbf{s}_i$ belongs to at least one subspace S_{j_i} in the collection, the scaled vector $a\mathbf{s}_i$ also belongs to S_{j_i} (it is a subspace, after all). So the vector

$$a\mathbf{v} = (a\mathbf{s}_1) + \cdots (a\mathbf{s}_n)$$

is a finite sum of vectors each of which is in $\cup_{i\in\Lambda} S_i$. This shows that $a\mathbf{v}$ belongs to T.

Since $T = \sum_{i\in\Lambda} S_i$ is closed under addition and scalar multiplication (and, taking $n = 1$, $\mathbf{s}_1 = \mathbf{0}$, it obviously contains the zero vector so is non-empty), we see that this is indeed a subspace using Theorem 1.1.

> $\star$ Show that $\sum_{i\in\Lambda} S_i$ is the least upper bound of the set $\{S_i\}_{i\in\Lambda}$ where subspaces are ordered under inclusion.

We must show that this is a subspace containing all of the subspaces S_i (which is evident); that there is a unique "smallest" subspace containing all of the S_i; and that this is the one.

First note that the definition of T allows us to choose any $\mathbf{s} \in S_j$ for any $j \in \Lambda$ and select $\mathbf{v} = \mathbf{s}$. So $S_j \subseteq T$ for every $j \in \Lambda$. This proves that T is indeed an upper bound for the collection $\{S_j \mid j \in \Lambda\}$.

Now to prove that it is the "smallest" subspace containing all of the S_j, we simply prove that it contains any subspace W of V with $\cup_{j\in\Lambda} S_j \subseteq W$. Let W be any such subspace. Let $\mathbf{v}$ be any vector in T. Then there exist vectors $\mathbf{s}_1, \ldots, \mathbf{s}_n$ chosen from $\cup_{j\in\Lambda} S_j$ such that $\mathbf{v} = \mathbf{s}_1 + \cdots + \mathbf{s}_n$. By hypothesis, each of these vectors $\mathbf{s}_i$ belongs to W since W contains the subspace it is chosen from (among those indexed by Λ). Since W is a subspace, W must contain the sum $\mathbf{v} = \mathbf{s}_1 + \cdots + \mathbf{s}_n$ of these vectors; so $\mathbf{v} \in W$. This proves that $T \subseteq W$.

Since T is a subspace containing all of the S_j, AND T is contained in any subspace W that contains all S_j, T is indeed the "smallest", the least upper bound of the set $\{S_i\}_{i\in\Lambda}$, where subspaces are ordered by inclusion. $\square$

2. Roman p57, Question 17

An *affine subspace* is a subset of vector space V of the form $\mathbf{v} + S$ for some vector subspace S of V, where $\mathbf{v} + S = \{\mathbf{v} + \mathbf{s} \mid \mathbf{s} \in S\}$.

$\star$ (a) An affine subspace $\mathbf{v} + S$ is a subspace of V if and only if $\mathbf{v} \in S$.

($\Rightarrow$) Assume S is a subspace of V and $\mathbf{v} \in V$ such that $\mathbf{v} + S$ is again a subspace of V. Then $\mathbf{v} + S$ contains the zero vector, so there is some $\mathbf{w} \in S$ such that

$$\mathbf{v} + \mathbf{w} = \mathbf{0}.$$

Clearly $\mathbf{w} = -\mathbf{v}$. So S contains $-\mathbf{v}$ and, as S is closed under scalar multiplication, S contains $(-1)\mathbf{w} = (-1)(-\mathbf{v}) = \mathbf{v}$ as well.
($\Leftarrow$) On the other hand, if S is a subspace and $\mathbf{v} \in S$, then $\mathbf{v} + S = S$ is again a subspace. To see this, note that every element $\mathbf{s} \in S$ is uniquely expressible as $\mathbf{s} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{w} \in S$ (namely $\mathbf{w} = \mathbf{s} - \mathbf{v}$). So whenever $\mathbf{v}$ belongs to S, the affine subspace is a subspace; in fact it is just S.

$\star$ (b) Any two affine subspaces of the form $\mathbf{v} + S$ and $\mathbf{w} + S$ (same S) are either equal or disjoint.

Suppose that $\mathbf{v} + S$ is not disjoint from $\mathbf{w} + S$. Then they have a vector in common. Let $\mathbf{u} \in (\mathbf{v} + S) \cap (\mathbf{w} + S)$. Then there exist vectors $\mathbf{v}'$ and $\mathbf{w}'$ in S with

$$\mathbf{u} = \mathbf{v} + \mathbf{v}', \qquad \mathbf{u} = \mathbf{w} + \mathbf{w}'.$$

But then we have $\mathbf{v} - \mathbf{w} \in S$ since

$$\mathbf{v} - \mathbf{w} = \mathbf{w}' - \mathbf{v}' \in S.$$

So $\mathbf{w}$ belongs to $\mathbf{v} + S$, say $\mathbf{w} = \mathbf{v} + \mathbf{s}$, and

$$\mathbf{w} + S = (\mathbf{v} + \mathbf{s}) + S = \mathbf{v} + (\mathbf{s} + S) = \mathbf{v} + S$$

since $\mathbf{s} + S = S$. (Check this on your own!)

3. Prove that every bijective linear transformation is a vector space isomorphism.

Let $\tau : V \to W$ be a bijective linear transformation. Since τ is a bijection, we know that τ^{-1} is a function: it is defined by $\tau^{-1}(\mathbf{w}) = \mathbf{v}$ if and only if $\tau(\mathbf{v}) = \mathbf{w}$.

Let $\tau^{-1}(\mathbf{w}_1) = \mathbf{v}_1$ and $\tau^{-1}(\mathbf{w}_2) = \mathbf{v}_2$. Then, by definition, $\tau(\mathbf{v}_1) = \mathbf{w}_1$ and $\tau(\mathbf{v}_2) = \mathbf{w}_2$. Since τ is a linear transformation, we have

$$\tau(\mathbf{v}_1 + \mathbf{v}_2) = \tau(\mathbf{v}_1) + \tau(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$$

so that

$$\tau^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v}_1 + \mathbf{v}_2 = \tau^{-1}(\mathbf{w}_1) + \tau^{-1}(\mathbf{w}_2).$$

Likewise, if $\tau^{-1}(\mathbf{w}) = \mathbf{v}$ and $a \in \mathbb{F}$, then $\tau(\mathbf{v}) = \mathbf{w}$, $\tau(a\mathbf{v}) = a\tau(\mathbf{v}) = a\mathbf{w}$ giving us

$$\tau^{-1}(a\mathbf{w}) = a\mathbf{v} = a\tau^{-1}(\mathbf{w}). \quad \square$$

4. Prove that every matrix may be uniquely expressed as a sum of a symmetric matrix and skew-symmetric matrix. Express this statement as a direct sum decomposition of the space.

Proof: Existence:
Assume $1 + 1 \neq 0$ and, given $M \in \mathcal{M}_n(\mathbb{F})$, define $A = \frac{1}{2}(M + M^\top)$, and $B = \frac{1}{2}(M - M^\top)$. Then, $M = A + B$ and $A = A^\top$, and $-B = B^\top$.

Uniqueness:
Suppose that $\exists A_1, A_2$ symmetric, and B_1, B_2 skew-symmetric, and $M = A_1 + B_1 = A_2 + B_2$. Then, defining $C = A_1 - A_2$ we find $C = B_2 - B_1$. Since the A_j are symmetric, we have

$$C^\top = (A_1 - A_2)^\top = A_1^\top - A_2^\top = A_1 - A_2 = C$$

and yet, since the B_j are skew-symmetrix, we have

$$C^\top = (B_1 - B_2)^\top = B_1^\top - B_2^\top = -B_1 + B_2 = -C.$$

So $C^\top = C$ and $C^\top = -C$ forcing $C = -C$. Since $1 + 1 \neq 0$, this forces C to be the zero matrix. So $A_1 = A_2$, $B_1 = B_2$ and the decomposition is unique. □

Let $\mathcal{S}_n(\mathbb{F})$ denote the subspace of symmetric $n \times n$ matrices and let $\mathcal{SS}_n(\mathbb{F})$ denote the subspace of skew-symmetric $n \times n$ matrices. We have shown

$$\mathcal{M}_n(\mathbb{F}) = \mathcal{S}_n(\mathbb{F}) \oplus \mathcal{SS}_n(\mathbb{F}).$$

5. Determine all subsets of $\mathcal{S}$ which forms bases for W.

(a) $W = \left\{a\cos^2 t + b\sin^2 t + ce^t \mid a,b,c \in \mathbb{R}\right\}$

bases: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

(b) $W = \left\{at^2 + b \mid a,b \in \mathbb{Q}\right\}$

bases: $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_4\}$, $\{\mathbf{v}_2, \mathbf{v}_4\}$

(c) $W = \mathcal{M}_{2,2}(\mathbb{Z}_2)$

bases: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.