MA3825 Rings & Fields
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December 2, 2009 NAME: ______________________________

Algebra Assignment 5

DUE DATE: Monday, December 8, 2009, by 4:30pm in my office mail slot.

In the interest of time, I am giving this one multiple choice assignment. The goal is to get you to read the book carefully and to meditate on the fundamental concepts presented there. (It is also my goal to develop critical thinking skills and professional technical writing skills, but these will be addressed in other exercises.)

Please put your name at the top of this paper and, after careful thought, simply circle the correct answer to each of the eight problems below. Any answer which is ambiguous will be assumed to be incorrect.

1. Consider the following statements about a ring R and its corresponding polynomial ring:

   (I) If R is an integral domain, then $R[x]$ is an integral domain

   (II) If R is a PID, then $R[x]$ is a PID.

   (III) If R is a field, then $R[x]$ is a field

   (IV) If R is a field, then $R[x]$ is a PID.

   Which of the above statements is true?

   (a) none

   (b) (I) and (II) only

   (c) (I) and (IV) only

   (d) (I), (II) and (III)

   (e) all

2. In $\mathbb{Z}_{10}[x]$, the product of $f(x) = 4x^3 + 5x^2 + 7x + 2$ and $g(x) = 5x^3 + x^2 + 5x + 5$ is

   (a) $20x^6 + 29x^5 + 60x^4 + 57x^3 + 62x^2 + 45x + 10$

   (b) $4x^5 + 5x^4 + 5x$

   (c) $4x^5 + 2x^3 + 2x^2$

   (d) $9x^5 + 2x^3 + 2x^2 + 5x$

3. If $\mathbb{F}$ is any field, then a *greatest common divisor* (gcd) of $f(x)$ and $g(x)$ in $\mathbb{F}[x]$ can be found by repeatedly applying the Division Algorithm, exactly as we do in the case of the integers: the gcd is the last non-zero remainder. But it is seldom unique in $\mathbb{F}[x]$: if $h(x)$ is a gcd for $f(x)$ and $g(x)$, then so also is $ch(x)$ for any nonzero $c \in \mathbb{F}$.

   Which of the following is a gcd for

$$f(x) = 2x^5 - 3x^4 + 4x^3 + 4x^2 + 2x + 7 \qquad \text{and} \qquad g(x) = 2x^4 - 7x^3 + 23x - 42$$

   in $\mathbb{Q}[x]$?

   (a) 1 (i.e., $f(x)$ and $g(x)$ are relatively prime)

   (b) $\frac{400}{81}x^2 - \frac{1000}{81}x - \frac{1400}{81}$

   (c) $-6x^2 + 15x - 21$

   (d) $2x - 5$

4. Which of the following polynomials is irreducible in $\mathbb{Q}[x]$?

   (a) $a(x) = x^2 + 20x + 91$

   (b) $b(x) = 21x^4 + 42x^2 - 840x + 98$

   (c) $c(x) = 23$

   (d) $d(x) = x^4 + 5x^3 + 19x^2 + 30x + 36$

5. Let p be a prime and let $f(x) = x^2 + ax + b$ in $\mathbb{Z}_p[x]$. Which of the following conditions guarantees that $f(x)$ is irreducible in $\mathbb{Z}_p[x]$?

   (a) $a = 0$

   (b) $b = 0$

   (c) b is a non-square in $\mathbb{Z}_p$

   (d) $a = 0$ and $p - b$ is a non-square in $\mathbb{Z}_p$

6. In the text, we see that $x^4 + 1$ is reducible modulo p for every prime p. Which of the following polynomials in $\mathbb{Z}[x]$ is reducible over $\mathbb{Z}_p$ for every prime p?

   (a) $a(x) = x^4 + 2$

   (b) $b(x) = x^4 + 3$

   (c) $c(x) = x^4 + x^2 + 1$

   (d) $d(x) = x^4 + 2x^2 + 2$

7. Which of the following factor rings is a field:

(a) $\mathbb{Z}_3[x]/\langle x^2+2\rangle$

(b) $\mathbb{Z}_3[x]/\langle x^2+x+1\rangle$

(c) $\mathbb{Z}_3[x]/\langle x^3+2x+2\rangle$

(d) $\mathbb{Z}_3[x]/\langle x^3+x+2\rangle$

8. Let D be an integral domain. Consider three binary relations on D:

(I) $a \sim b$ if ab is a unit

(II) $a \sim b$ if a and b are associates

(III) $a \sim b$ if $a-b$ is irreducible

Which of the above relations is an equivalence relation on D?

(a) (I) only

(b) (I) and (II) only

(c) (II) only

(d) (I) and (III) only

(e) all