MA3823 Group Theory
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Group Theory Assignment 3

Due Date: Monday December 3, 2012, by 4:30pm in my office mail slot. The faculty mailboxes are in the Department Office, SH108.

Please recall the presentation rules for assignments in this course; they are reproduced at the end of this document for your convenience.

1. Let G be a group and let $H \leq G$. For each $a \in H$ define the *conjugacy class containing* a in each of these groups

$$\mathrm{cl}_G(a) = \left\{ x^{-1}ax \mid x \in G \right\}$$
$$\mathrm{cl}_H(a) = \left\{ x^{-1}ax \mid x \in H \right\}$$

   (a) Prove: For every $a \in H$, $\mathrm{cl}_H(a) \subseteq \mathrm{cl}_G(a)$.

   (b) Give an example where $\mathrm{cl}_H(a) \neq \mathrm{cl}_G(a)$. Explain. (Bonus: Can you choose your example so that $\mathrm{cl}_G(a) \subseteq H$?)

2. Give a group-theoretic proof that $(\mathbb{Q}, +)$ is not isomorphic to $(\mathbb{R}^+, \cdot)$.

3. Let G be a group and let $\alpha : G \to G$ be an automorphism. Prove that $\{x \in G \mid \alpha(x) = x\}$ is a subgroup of G.

4. Let G be a group and let $g, h \in G$ with corresponding inner automorphisms φ_g and φ_h, respectively. Prove that $\varphi_g = \varphi_h$ (as functions) if and only if $hg^{-1} \in Z(G)$, the center of G.

5. Prove that every automorphism of the rational numbers (under addition) takes the form $\varphi(x) = kx$ for some rational number $k \neq 0$.

6. Let $\varphi : G \to H$ be a group homomorphism with image $\varphi(G)$. Prove: If G is Abelian, then $\varphi(G)$ is an Abelian subgroup of H.

7. Let G be a group, let H and K be subgroups of G and $g \in G$. Prove that $g(H \cap K) = gH \cap gK$. (Be careful here!)

8. Let a and b be nonidentity elements of different orders in a group G of order 145. Prove that the only subgroup of G that contains both a and b is G itself.

9. Recall that, for any integer n, $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n. Prove that, if a is any positive integer relatively prime to n then $a^{\phi(n)} \bmod n = 1$.

10. **(a)** Find all the conjugacy classes of the dihedral groups D_4 and D_5. In our classroom notation

$$D_4 = \{e,\ r,\ r^2,\ r^3,\ s,\ sr,\ sr^2,\ sr^3\};$$

these correspond to Gallian's cumbersome notation as follows[1]:

$$e = R_0,\ r = R_{90},\ r^2 = R_{180},\ r^3 = R_{270},\quad s = V,\ sr = D',\ sr^2 = H,\ sr^3 = D.$$

(b) Prove: For all n, every conjugacy class in the dihedral group D_n has size one or two.

PRESENTATION RULES FOR MA3823 ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only one side of each sheet of paper; **Any paper submitted which is sloppy or uses two sides of a page will be returned immediately with no credit.**

III) Show your work. Explain your answers using FULL SENTENCES;

IV) No late assignments will be accepted for credit.

[1] To be clear, other valid bjections can be given