MA3823 Group Theory
W. J. Martin
October 23, 2012

Group Theory Assignment 1

Due Date: Tuesday October 30, 2012, by 4:30pm in my office mail slot. The faculty mailboxes are in the Department Office, SH108.

Please carefully read the presentation rules below. The problem statements begin on the next page.

BASIC RULES FOR MA3823 ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only one side of each sheet of paper; **Any paper submitted which is sloppy or uses two sides of a page will be returned immediately with no credit.**

III) Show your work. Explain your answers using FULL SENTENCES;

IV) No late assignments will be accepted for credit.

Please provide concise, rigorous proofs of the following ten statements.

1. Let S be a set and let $\sim$ and $\equiv$ be two equivalence relations on S. Then their intersection $\sim \cap \equiv$ is also an equivalence relation on S. (Let's call this last relation $\cong$ in our proof.)

2. Let G be a group and let $a \in G$. For a positive integer n, define $a^{-n} := (a^{-1})^n$ (as on p51). For all positive integers n, we have $a^{-n} = (a^n)^{-1}$. (Prove this by induction.)

3. Let G be a group and let $a, b \in G$. For any positive integer n, we have $(a^{-1}ba)^n = a^{-1}b^n a$. (Prove this by induction.)

4. The set of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ with entries from $\mathbb{Q}$ and satisfying $a \neq 0$, $d \neq 0$ forms a group under ordinary matrix multiplication. (Over what other number systems, besides $\mathbb{Q}$, is this conclusion still valid?)

5. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{N}\}$. Then G is a group under multiplication of complex numbers but does not form a group under addition of complex numbers.

6. Let G be a group and $a, b \in G$. Then $(ab)^2 = a^2b^2$ if and only if $ab = ba$.

7. If G is a group in which $a^2 = e$ for all $a \in G$, then G is Abelian.

8. If G is a group and for all $a, b, c \in G$, $ab = ca$ implies $b = c$, then G is Abelian.

9. **(a)** Give an example of a group G and elements $a, b, c, d, x \in G$ such that $axb = cxd$ and yet $ab \neq cd$. (I.e., "middle cancellation" is not valid in groups.)

   (b) Suppose group G enjoys the property that, for all $a, b, c, d, x \in G$, if $axb = cxd$ then $ab = cd$. Then G is Abelian.

10. Is the union of two groups always a group? Explain clearly in one or two paragraphs, using at least two examples.