Practice Problems – Convexity

An Assortment of Problems in Convex Analysis

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1. Isabella has to decide which among the following sets is convex. You have kindly offered to help her. But first ...

   (a) Let S be a subset of the Euclidean plane R^2. What, precisely, does it mean to say that S is "convex".

   (b) For each of the following sets S, determine whether or not S is convex. If not, give a convincing and precise argument that it is not convex.

   (i) $S = \{(x,y) \in \mathbb{R}^2 \ : \ x + y \leq 7 \text{ and } x > 2y + 1\}$

   (ii) $S = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \geq 16\}$

   (iii) $S = \{(x,y) \in \mathbb{R}^2 \ : \ -2 \leq x \leq 3 \ \text{ and } y^2 = 1\}$

   (iv) $S = \{(x,y) \in \mathbb{R}^2 \ \mid 2x + y \leq 20 \ \text{ and } y > x^2 + 5\}$

   (v) $S = \mathbb{R}^2$

2. Show, by concrete example, that it is possible to have two convex sets C_1 and C_2 whose union $C = C_1 \cup C_2$ is <u>not</u> convex.

3. Tamara is considering a linear programming problem in standard form:

$$\begin{array}{rll} \textbf{max} & c^T x & \\ \textbf{subject to} & Ax & \leq b \\ & x & \geq 0 \end{array}$$

   Prove for her that the feasible region

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \leq b, \ x \geq 0\}$$

   is a convex set. (Be complete and be mathematically precise.)

4. Let S be a non-empty subset of $\mathbb{R}^n$. Prove that $\text{conv}(S)$ is equal to the set of all convex combinations of points from S.

5. State and prove Carathéodory's Theorem.

6. State and prove the Separation Theorem for polyhedra.

7. Let $\mathcal{C} = \{C_j \mid j = 1, 2, 3, \ldots\}$ be a (possibly infinite) collection of convex subsets of $\mathbb{R}^n$. Let S be the intersection of all these sets (i.e., S includes just those points of $\mathbb{R}^n$ that belong to every single C_j). Prove that $S = \cap_j C_j$ is convex.

8. Prove that a linear function on a line segment in $\mathbb{R}^n$ is maximized at (at least) one of the endpoints.

9. Prove that a linear function $f(\mathbf{x}) = c^\top \mathbf{x}$ defined on a bounded convex set C achieves its maximum on the boundary of C: that is, prove that there is some $\mathbf{y} \in C$ such that $\mathbf{y} \notin \text{conv}(C - \{\mathbf{y}\})$ with $c^\top \mathbf{y} \geq c^\top \mathbf{x}$ for all $\mathbf{x}$ in C. [This is hard.]