MA3231 Linear Programming
W. J. Martin
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LP Assignment 5

DUE DATE: Tuesday, September 25, 2018, at the beginning of class.

Please carefully review the assignment presentation rules on the back of this sheet.

Provide neat and careful solutions to the following five problems:

1.) Your company uses three resources — snot, spit and filth — to manufacture three products — goodness, compassion and cheer. Let

- x_1 denote the number of expressions of goodness to be produced;
- x_2 denote the number of acts of compassion to be produced;
- x_3 denote the number of bundles of cheer to be produced.

With 120 and 240 gallons, respectively, of snot and spit and and 180 tons of filth available, the following LP formulation uses per-unit profit for each product in order to maximize overall profit (in euros) subject to resource constraints:

$$\begin{array}{rl} \textbf{maximize} & 20x_1 + 30x_2 + 40x_3 \\ \textbf{subject to} & x_1 + x_2 + x_3 \leq 120 \\ & 2x_1 - x_2 + 3x_3 \leq 240 \\ & 2x_1 + 4x_2 \leq 180 \\ & x_1, \; x_2, \; x_3 \geq 0 \end{array}$$

The optimal dictionary is

$$\begin{array}{ccrrrr} \zeta & = & 4500 & -\frac{35}{2}x_1 & -\frac{65}{2}x_4 & -\frac{5}{2}x_5 \\ \hline x_2 & = & 30 & -\frac{1}{4}x_1 & -\frac{3}{4}x_4 & +\frac{1}{4}x_5 \\ x_3 & = & 90 & -\frac{3}{4}x_1 & -\frac{1}{4}x_4 & -\frac{1}{4}x_5 \\ x_6 & = & 60 & - x_1 & +3x_4 & - x_5 \end{array}$$

(a) Write down the shadow prices (optimal dual variable values) for each of the three resources. For each, write a sentence explaining what this means in terms of the manufacturing problem.

(b) Suppose that an extra gallon of snot becomes available to you. How much would you be willing to pay for it and why? Up to what limit is this price sensible? Do the same for spit and filth. (Do not pivot.)

(c) Suppose that the "per-expression" profit for goodness changes from \$20 to \$35. Would you start producing goodness or not? Explain. (Do not pivot.)

(d) Next, suppose that, after optimizing, you learn that the profit per unit for cheer has diminished from 40 to 10. Write down the modified final dictionary. If it is no longer optimal, then pivot to optimality.

(e) Suppose that, after solving the original LP, the amount of spit available is reduced from 240 gallons to 160 gallons. Work out the modified final dictionary. Apply the dual simplex method to pivot from this dictionary to optimality.

(f) What is the smallest number of gallons b_2 of spit for which the basis $\{2, 3, 6\}$ is optimal?

2.) Consider the LP

$$\begin{array}{rl} \textbf{maximize} & 4x_1 + 2x_2 + 3x_3 \\ \textbf{subject to} & x_1 + x_2 + x_3 \leq 14 \\ & 4x_1 + x_2 + 2x_3 \leq 40 \\ & 2x_1 + x_2 + x_3 \leq 21 \\ & x_1, \ x_2, \ x_3 \geq 0 \end{array}$$

with optimal dictionary

$$\begin{array}{rcrrrr} \zeta & = & 48 & -\frac{1}{2}x_2 & -2x_4 & -\frac{1}{2}x_5 \\ \hline x_3 & = & 8 & -\frac{3}{2}x_2 & -2x_4 & +\frac{1}{2}x_5 \\ x_6 & = & 1 & -\frac{1}{2}x_2 & & +\frac{1}{2}x_5 \\ x_1 & = & 6 & +\frac{1}{2}x_2 & +x_4 & -\frac{1}{2}x_5 \end{array}$$

(a) For each variable (decision variables as well as slack variables), find the range of optimality for the current basis relative to objective coefficient c_j. That is, for what real numbers c_j is the current dictionary still optimal? (Be sure to give your answer in terms of the original data, c_j, not relative to the value $\bar{c}_j$ appearing in this dictionary.)

(b) For decision variables x_1, x_2 and x_3, tell me what must be done next (a simplex pivot or a dual simplex pivot?) when optimality is comprised. **Do not pivot**. Simply give the entering and leaving variable.

(c) Compating the original problem and the optimal dictionary above, for each entry of vector $\mathbf{b} = (14, 40, 21)$, find the range of optimality for the current basis relative to right-hand side value b_i. For what real numbers b_i is the (slightly modified) current dictionary still optimal? When optimality is compromised, tell me what must be done next (a simplex pivot or a dual simplex pivot?). **Do not pivot**, but be sure to give the entering and leaving variable for the first pivot on your way back to optimality.

3.) Apply the Dual Simplex Method on the following problem.

$$\begin{array}{lrcrcrcr}
\textbf{max} & -12\,x_1 & - & 8\,x_2 & - & 12\,x_3 & & \\
\textbf{s.t.} & -2\,x_1 & - & x_2 & - & x_3 & \leq & -1 \\
 & -\,x_1 & - & x_2 & - & 2\,x_3 & \leq & -3 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}$$

4.) Consider the LP problem

$$\begin{array}{ll}
\textbf{maximize} & -4x_1 - 4x_2 - 15x_3 \\
\textbf{subject to} & x_1 - x_2 - 6x_3 \quad \leq -2 \\
 & -3x_1 + 2x_2 + 14x_3 \quad \leq -6 \\
 & x_1, \;\; x_2, \;\; x_3 \qquad \geq 0
\end{array}$$

(a) Optimize this problem **using the dual simplex method**.

(b) Write down the dual of this LP.

(c) Optimize this dual problem using the ordinary simplex method. (NOTE: Recall that our simplex algorithm only maximizes, so use the fact that **min** ω has the same effect as **max** $-\omega$ except that the answer has its sign reversed.)

5.) Suppose $\mathcal{B}$ is the optimal basis for an LP problem with n variables and m constraints in standard form $\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Consider now, instead, the region of all vectors $\mathbf{b}'$ for which $\mathcal{B}$ is still the optimal basis for the modified problem

$\max \mathbf{c}^\top \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}'$, $\mathbf{x} \geq \mathbf{0}$.

Now, for this inverted problem — where A, $\mathbf{c}$ and $\mathcal{B}$ are fixed, $\mathbf{b}' \in \mathbb{R}^m$ is varying, and $\mathbf{x}$ given by the dictionary corresponding to basis $\mathcal{B}$ — determine whether or not the objective function is bounded. Can you give an example where $\mathbf{c}^\top \mathbf{x}$ is bounded? Can you give an example where $\mathbf{c}^\top \mathbf{x}$ is unbounded? Why or why not? Explain.

BASIC RULES FOR LP ASSIGNMENTS

I) Each student must compose his/her assignments independently. However, rough work may be done in groups;

II) Write legibly and use only one side of each sheet of paper;

III) Show your work. Explain your answers using FULL SENTENCES;

IV) Late assignments will not, in general, be accepted for credit.