MA3231 Linear Programming
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LP Assignment 4

DUE DATE: Thursday, September 20th, 2018, at the beginning of class.

Please be sure to recall and observe Dr. Martin's assignment presentation rules for this course.

Provide neat and careful solutions to the following five problems:

1.) In this exercise, we simply work on our notation. Suppose we are given the following linear programming problem:

$$\begin{array}{rl} \textbf{maximize} & x_2 + x_3 \\ \textbf{subject to} & x_1 + x_2 + x_3 \leq 8 \\ & x_1 + 2x_2 + x_3 \leq 10 \\ & x_1 - x_2 + 5x_3 \leq 28 \\ & x_1, \ x_2, \ x_3 \geq 0 \end{array}$$

After introducing slack variables x_4, x_5, x_6 and applying the simplex method, we arrive at the optimal dictionary

$$\begin{array}{ccccccc} \zeta & = & 8 & -x_1 & -x_4 & \\ \hline x_3 & = & 6 & -x_1 & -2x_4 & +x_5 \\ x_2 & = & 2 & & +x_4 & -x_5 \\ x_6 & = & 0 & +4x_1 & +11x_4 & -6x_5 \end{array}$$

For this dictionary, write down the basis $\mathcal{B}$ and the set of non-basic variables $\mathcal{N}$. Write down the partition $A = [B|A_{\mathcal{N}}]$ of the original constraint matrix corresponding to this partition of variables. Write down the matrix B^{-1}, the vector b, and compute their product.

2.) Find the dual problem to the following LP problem:

$$\begin{array}{rl} \textbf{minimize} & 11x_1 + 12x_2 + 13x_3 + 14x_4 + 15x_5 \\ \textbf{subject to} & 2x_2 + 4x_4 \geq 1 \\ (\mathsf{P}) & x_1 + x_2 + x_3 + x_4 + x_5 = 22 \\ & x_1 + 2x_2 + 3x_3 - 4x_4 - 5x_5 \leq 333 \\ & x_1 \geq 0, \ x_2 \geq 0, \quad x_3, \quad x_4, \quad x_5 \quad \text{unrestricted} \end{array}$$

Explain each feature with a few words.

3.) Consider the LP

$$\begin{array}{lrl} \textbf{minimize} & 20x_1 + 12x_2 + 36x_3 + 42x_4 & \\ \textbf{subject to} & x_1 + \quad x_2 + \quad 2x_3 + \quad x_4 & \geq 1 \\ \text{(P)} & x_1 \qquad\quad + \quad x_3 + \ 3x_4 & \geq 2 \\ & x_1, \ \ x_2, \ \ x_3, \ \ x_4 & \geq 0 \end{array}$$

(a) Carefully plot the feasible region for the **dual** problem.
(b) Solve this dual LP by the graphical method.
(c) From the picture, write down the optimal basis and the optimal solution for the primal problem (P) above.

4.) **(a)** Find the dual problem to the following LP problem:

$$\begin{array}{lrl} \textbf{maximize} & 2x_1 + 3x_2 & \\ \textbf{subject to} & x_1 - 2x_2 & \leq 9 \\ \text{(P}') & 4x_1 + 3x_2 & \leq 15 \\ & -x_1 + 2x_2 & \leq 15 \\ & x_1, \ \ x_2 & \geq 0 \end{array}$$

(b) Applying the Complementary Slackness Conditions, determine, for each of the following vectors $\mathbf{x}$ whether $\mathbf{x}$ is <u>INFEASIBLE</u>, <u>FEASIBLE BUT NOT OPTIMAL</u>, or <u>OPTIMAL</u> for the above problem (P′):

(i) $\mathbf{x} = (1, 8)$, *(ii)* $\mathbf{x} = (0, 5)$, *(iii)* $\mathbf{x} = (2, 2)$, *(iv)* $\mathbf{x} = (3, 1)$.

5.) Use the Strong Duality Theorem to prove the following theorem:

Let A be an $m \times n$ matrix and let $\mathbf{b}$ denote some vector of length m. Then either there is a non-negative vector $\mathbf{x}$ of length n such that $A\mathbf{x} = \mathbf{b}$ or there is a vector $\mathbf{y}$ of length m such that all entries of $\mathbf{y}^\top A$ are non-negative and yet $\mathbf{y}^\top \mathbf{b} < 0$. Only one of these two alternatives can hold at a time.

(More compactly, we can write this as: given A and $\mathbf{b}$, either $\exists \mathbf{x} \geq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{b}$ or $\exists \mathbf{y}$ such that $\mathbf{y}^\top A \geq \mathbf{0}$ and $\mathbf{y}^\top \mathbf{b} < 0$, NOT BOTH.)