C Term, Sections C01-C04
Date: February 17, 2010

Instructor: W J Martin

type b

Matrices and Linear Algebra I – Test 2

NAME: SAMPLE SOLUTIONS

Time: 110 minutes.

SECTION:
C01 (Erin, noon) C02 (Alyssa, 1pm) C03 (Grant, 2pm) C04 (Nghia, 3pm)

WRITE YOUR NAME IN THE SPACE ABOVE LABELED "NAME". CIRCLE YOUR SECTION.

Important Instructions: This test consists of 4 questions. Please answer all questions right on these pages, beginning each answer just below the statement of the question. If you need extra space (or space for rough work), use the back of the page, clearly indicating to which question the work pertains. **Show your work!** An incorrect answer with no explanation will receive no credit. A correct answer with no rough work will receive only partial credit. c

Please note that no texts, notes, or scrap paper are permitted.

As well, please note that **no electronic devices** are permitted to be out in the open during the examination. Any use of electronic devices is considered cheating. Any communication between examinees during the examination is likewise an act of academic dishonesty.

DO NOT OPEN THIS TEST BOOKLET UNTIL SO INSTRUCTED.

This grid for graders' use only.

Question	1	2	3	4	Total (out of 40)
Score	10	10	10	10	40

1.) Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x,y) = (2x + 4y, x - 2y), \qquad S(x,y) = (-2x, \frac{y}{2}).$$

(a) [*3 points*] Find the standard matrices A and B for the transformations T and S. (Please be sure to use the name A for T and B for S.)

$$A = \left[T(\bar{e}_1) \,\middle|\, T(\bar{e}_2) \right] = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix} \qquad B = \left[S(\bar{e}_1) \,\vdots\, S(\bar{e}_2) \right] = \begin{bmatrix} -2 & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix}$$

(b) [*4 points*] Carefully compute the matrix $M = A^{-1}BA^{\top}$.

First invert A: $\quad [A \,\vdots\, I] = \left[\begin{array}{cc:cc} 2 & 4 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & -2 & 0 & 1 \\ 0 & 8 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & \tfrac{1}{4} & \tfrac{1}{2} \\ 0 & 1 & \tfrac{1}{8} & -\tfrac{1}{4} \end{array}\right]$

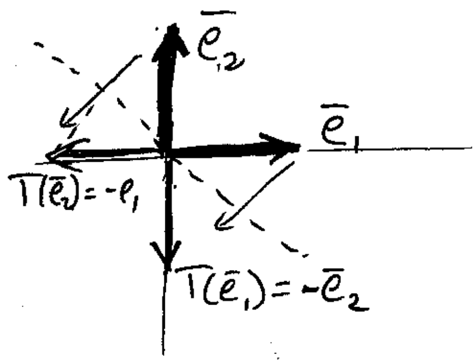
$$\begin{aligned} M &= A^{-1} B A^{T} \\ &= \begin{bmatrix} \tfrac{1}{4} & \tfrac{1}{2} \\ \tfrac{1}{8} & -\tfrac{1}{4} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix} A^{T} \\ &= \begin{bmatrix} -\tfrac{1}{2} & \tfrac{1}{4} \\ -\tfrac{1}{4} & -\tfrac{1}{8} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -2 \end{bmatrix} \end{aligned}$$

$$\boxed{M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$$

(c) [*3 points*] Describe geometrically (in words) the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose standard matrix is M.

Multiplication by M reflects each point across the line

$x + y = 0$



2.) In parts (a)–(c), compute the determinant, $\det A$.

(a) [*2 points*] $A = \begin{bmatrix} 75 & 13 & -2 \\ 11 & 2 & 0 \\ 15 & -9 & 9 \end{bmatrix}$

$$|A| = \left| \begin{matrix} 75 & 13 & -2 \\ 11 & 2 & 0 \\ 15 & -9 & 9 \end{matrix} \right. \begin{matrix} 75 & 13 \\ 11 & 2 \\ 15 & -9 \end{matrix} = 75\cdot 2\cdot 9 + \cancel{13\cdot 0\cdot 15} + (-2)\cdot 11\cdot(-9) - 15(2)(-2) - \cancel{(-9)0\cdot 75} - 9\cdot(11)\cdot(13)$$

$$\boxed{\det A = 321}$$

(b) [*2 points*] $A = \begin{bmatrix} 4 & -9 & 1 & 0 & 7 \\ 0 & -3 & 30 & 13 & 2 \\ 0 & 0 & 2 & -8 & 0 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$

Upper triangular

$$\det A = 4\cdot(-3)\cdot 2\cdot 2\cdot 9 = \boxed{-432}$$

(c) [*6 points*] $A = \begin{bmatrix} 3 & 2 & 2 & 1 & -1 \\ 3 & 2 & 2 & 1 & 0 \\ 3 & 2 & 3 & 1 & -1 \\ 3 & 2 & 2 & 2 & -1 \\ 3 & 2 & 2 & 1 & 180 \end{bmatrix}$ (Be sure to show your work.)

Let $\Delta = \det A$

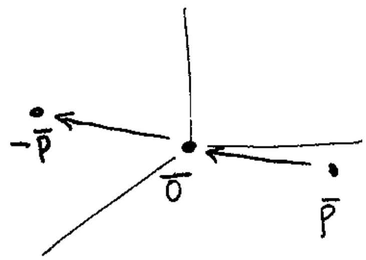
$$A \sim \begin{array}{c} \begin{bmatrix} 3 & 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 181 \end{bmatrix} \end{array} \begin{array}{l} R1 \\ R2 - R1 \\ R3 - R1 \\ R4 - R1 \\ R5 - R1 \end{array} \quad \det = \Delta$$

we can stop here: there is no pivot in the second column. So the matrix is singular and

$$\boxed{\det A = 0}$$

3.) Using homogeneous coordinates, find the linear transformations that achieve the following operations on 3-dimensional space.

(a) [*2 points*] Translate every (x, y, z) so that $(-2, 5, -3)$ is moved to the origin.



$h = 2$

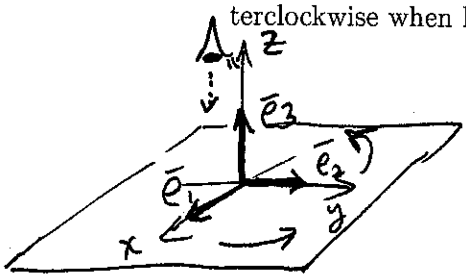
$K = -5$

$l = 3$

The matrix we need is

$$\left[\begin{array}{c|c} I_3 & \begin{matrix} h \\ K \\ l \end{matrix} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

(b) [*2 points*] Rotate every (x, y, z) by 90° around the z-axis. (Here the direction is counterclockwise when looking from $(0, 0, 10)$ toward the xy-plane.)



$T(\bar{e}_1) = \bar{e}_2$

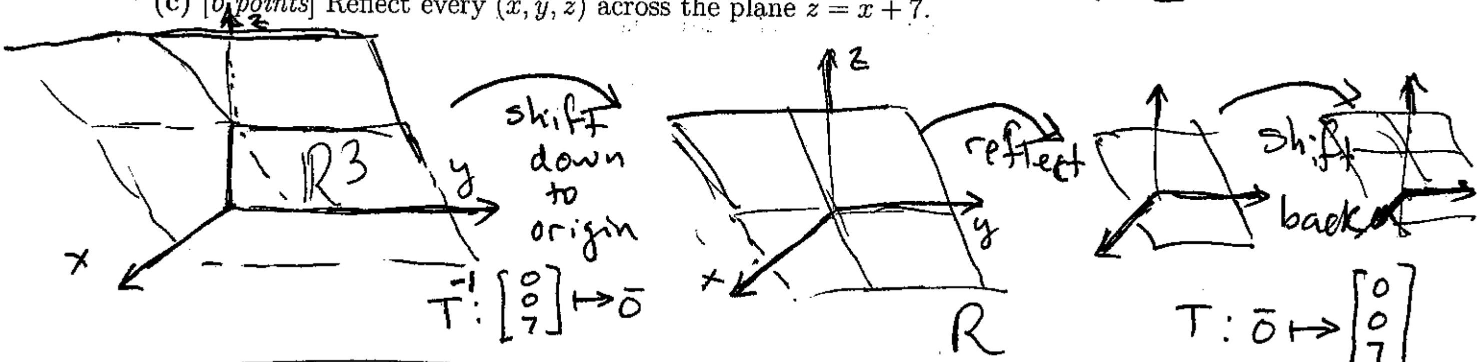
$T(\bar{e}_2) = -\bar{e}_1$

$T(\bar{e}_3) = \bar{e}_3$

So the matrix is

$$\left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

(c) [*6 points*] Reflect every (x, y, z) across the plane $z = x + 7$.



Reflection across plane $x = z$:

$R(\bar{e}_1) = \bar{e}_3 \quad R(\bar{e}_2) = \bar{e}_2 \quad R(\bar{e}_3) = \bar{e}_1$

$$R = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$M = TRT^{-1}$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 7 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$M = \begin{bmatrix} 0 & 0 & 1 & -7 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4(a) [*7 points*] Prove the following statement: If A is an $n \times n$ matrix and the columns of A form a linearly independent set, then the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one. Let A be an $n \times n$ matrix.

Assume the columns of A are linearly independent. Consider the function

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

via $T(\bar{x}) = A\bar{x}$

Suppose $\bar{x}$ and $\bar{y}$ are n-vectors and $T(\bar{x}) = T(\bar{y})$.

We need to show that $\bar{x} = \bar{y}$. Now write

$$A = \left[\bar{a}_1 \middle| \bar{a}_2 \middle| \cdots \middle| \bar{a}_n\right].$$

Then $T(\bar{x}) = A\bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n$

while $T(\bar{y}) = A\bar{y} = y_1\bar{a}_1 + y_2\bar{a}_2 + \cdots + y_n\bar{a}_n$.

So $T(\bar{x}) = T(\bar{y})$ gives us

$$x_1\bar{a}_1 + x_2\bar{a}_2 + \cdots + x_n\bar{a}_n = y_1\bar{a}_1 + y_2\bar{a}_2 + \cdots + y_n\bar{a}_n$$

$$(x_1 - y_1)\bar{a}_1 + (x_2 - y_2)\bar{a}_2 + \cdots + (x_n - y_n)\bar{a}_n = \bar{0}$$

Since the vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ are linearly independent, the only possibility is

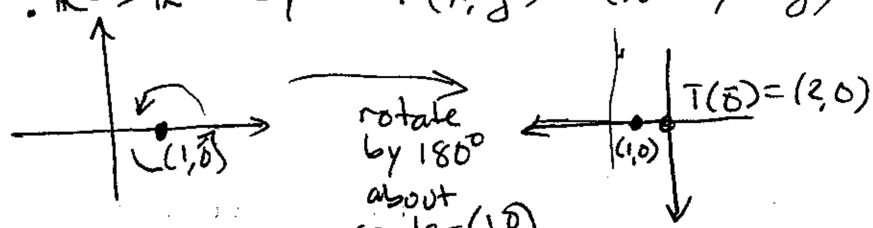
$$x_1 - y_1 = 0, \quad x_2 - y_2 = 0, \quad \ldots, \quad x_n - y_n = 0$$

so $x_1 = y_1$, $x_2 = y_2$, $\ldots$, $x_n = y_n$ and $\bar{x} = \bar{y}$.

Therefore we conclude that function T is one-to-one.

(b) [*3 points*] Give an example of a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is one-to-one and onto but is **not** a linear transformation. Explain.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (2 - x, -y)$



* one-to-one: If $T(a, b) = T(c, d)$

then $(2 - a, -b) = (2 - c, -d)$

so $a = c$ and $b = d$

* onto: given any (a, b) in the plane,

take $x = 2 - a$, $y = -b$ so that $T(x, y) = (a, b)$

* not linear: $T(2, 0) = (0, 0)$, $T(4, 0) = (-2, 0)$

so $T(2\bar{u}) \neq 2T(\bar{u})$ for $\bar{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$