C Term, Sections C01-C04
Date: February 17, 2010

Instructor: W. J. Martin
type a

Matrices and Linear Algebra I – Test 2

NAME: SAMPLE SOLUTIONS

Time: 110 minutes.

SECTION:
C01 (Erin, noon) C02 (Alyssa, 1pm) C03 (Grant, 2pm) C04 (Nghia, 3pm)

WRITE YOUR NAME IN THE SPACE ABOVE LABELED "NAME". CIRCLE YOUR SECTION.

Important Instructions: This test consists of 4 questions. Please answer all questions right on these pages, beginning each answer just below the statement of the question. If you need extra space (or space for rough work), use the back of the page, clearly indicating to which question the work pertains. **Show your work!** An incorrect answer with no explanation will receive no credit. A correct answer with no rough work will receive only partial credit. c

Please note that no texts, notes, or scrap paper are permitted.

As well, please note that **no electronic devices** are permitted to be out in the open during the examination. Any use of electronic devices is considered cheating. Any communication between examinees during the examination is likewise an act of academic dishonesty.

DO NOT OPEN THIS TEST BOOKLET UNTIL SO INSTRUCTED.

This grid for graders' use only.

Question	1	2	3	4	Total (out of 40)
Score	10	10	10	10	40

1.) Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = (x - \sqrt{3}y, y), \qquad S(x, y) = (\frac{x}{2}, 2y).$$

(a) [*3 points*] Find the standard matrices A, B and C for the transformations T, S and their composition $S \circ T$. (Please be sure to use the names A, B, C for these three matrices, in this order.)

$$A = \left[T(\bar{e}_1) \middle| T(\bar{e}_2) \right] = \begin{bmatrix} 1 & -\sqrt{3} \\ 0 & 1 \end{bmatrix} \qquad B = \left[S(\bar{e}_1) \middle| S(\bar{e}_2) \right] = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$C = BA = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 2 \end{bmatrix}$$ (Theorem in text)

(b) [*4 points*] Carefully compute the matrix $M = (A^\top)^{-1}BA$.

$$A^T = \begin{bmatrix} 1 & 0 \\ -\sqrt{3} & 1 \end{bmatrix}.$$

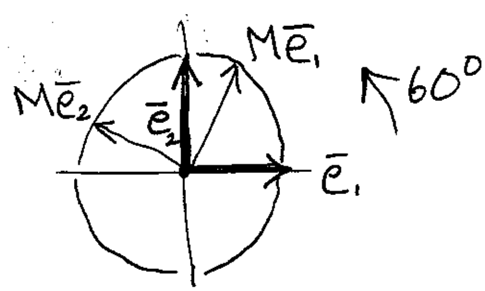
Need inverse: $\left[A^T \,\vdots\, I \right] = \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ -\sqrt{3} & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & \sqrt{3} & 1 \end{array}\right]$

$$(A^T)^{-1} = \begin{bmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{bmatrix}$$

$$M = (A^T)^{-1} BA = \begin{bmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

(c) [*3 points*] Describe geometrically (in words) the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose standard matrix is M.

M represents rotation by 60° counterclockwise about the origin.



2.) In parts (a)–(c), compute the determinant, $\det A$.

(a) [*2 points*] $A = \begin{bmatrix} 11 & 3 & -11 \\ 8 & -7 & 0 \\ 2 & -2 & -1 \end{bmatrix}$ (HINT: To simplify your arithmetic, collect terms with elevens in them before multiplying.)

$$|A| = \begin{vmatrix} 11 & 3 & -11 \\ 8 & -7 & 0 \\ 2 & -2 & -1 \end{vmatrix} \begin{matrix} 11 & 3 \\ 8 & -7 \\ 2 & -2 \end{matrix} = 11(-7)(-1) + \cancel{3\cdot 0\cdot 2} + (-11)\cdot 8\cdot(-2) - 2(-7)(-11) - \cancel{(-2)\cdot 0\cdot 11} - (-1)\cdot 8\cdot 3$$

$$= 11(7+16-14) + 24 = 99+24$$

$$\boxed{\det A = 123}$$

(b) [*2 points*] $A = \begin{bmatrix} 2 & -9 & 1 & 0 & 7 \\ 0 & -3 & 30 & 13 & 2 \\ 0 & 0 & 13 & -8 & 0 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

upper triangular

$$\det A = 2\cdot(-3)\cdot 13\cdot(-1)\cdot 3 = 13\cdot 18 = \boxed{234}$$

(c) [*6 points*] $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 6 & 4 & 5 \\ 1 & 2 & 3 & 119 & 5 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 1 & 3 & 4 & 5 \end{bmatrix}$ (Be sure to show your work.)

Define $\Delta = \det A$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 115 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R2 - R1 \\ R3 - R1 \\ R4 - R1 \\ R5 - R1 \end{matrix} \qquad \det = \Delta$$

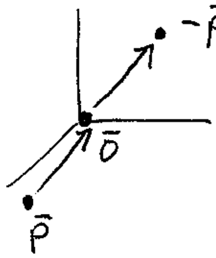
$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 115 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R5 \\ R2 \\ R3 \\ R4 \end{matrix} \quad \text{w/ 3 swaps} \qquad \det = -\Delta$$

$$\det = -345$$

$$\boxed{\det A = 345}$$

3.) Using homogeneous coordinates, find the linear transformations that achieve the following operations on 3-dimensional space.

(a) [*2 points*] Translate every (x, y, z) so that $(2, -3, -4)$ is moved to the origin.



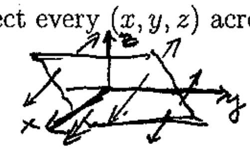
$h = -2,$
$k = 3$
$l = 4$

The matrix we want is

$$\left[\begin{array}{c|c} I & \begin{matrix} h \\ k \\ l \end{matrix} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

(b) [*2 points*] Reflect every (x, y, z) across the plane $x = z$.

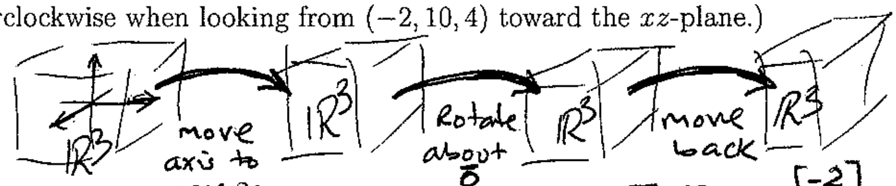
Need

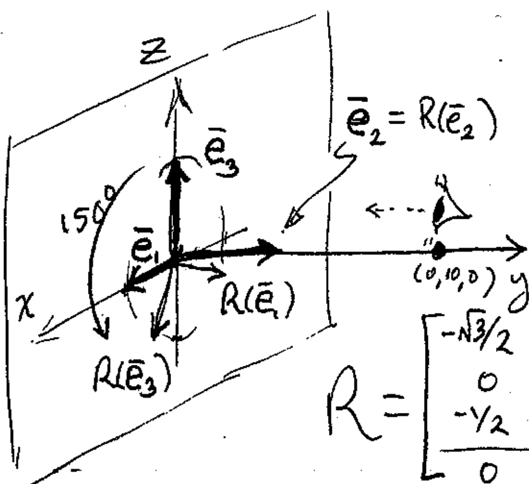


$$T(\bar{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \bar{e}_3, \quad T(\bar{e}_2) = \bar{e}_2, \quad T(\bar{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \bar{e}_1. \quad \text{So} \quad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

is the matrix we want.

(c) [*6 points*] Rotate every (x, y, z) by 150° about the line joining $(-2, 0, 4)$ to $(-2, 10, 4)$. (Here the direction is counterclockwise when looking from $(-2, 10, 4)$ toward the xz-plane.)





$$T^{-1}: \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \to \bar{o} \qquad R \qquad T: \bar{o} \mapsto \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

$$T = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$R = \left[\begin{array}{ccc|c} -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad T^{-1} = \left[\begin{array}{c|c} I_3 & \begin{matrix} 2 \\ 0 \\ -4 \end{matrix} \\ \hline \bar{o}^T & 1 \end{array}\right]$$

$$M = TRT^{-1} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -\sqrt{3}/2 & 0 & 1/2 & -2 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 & 4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{M = \left[\begin{array}{ccc|c} -\sqrt{3}/2 & 0 & 1/2 & -4-\sqrt{3} \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & -\sqrt{3}/2 & 3+2\sqrt{3} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}$$

4(a) [*7 points*] Prove the following statement: If A is an $n \times n$ matrix and the columns of A span $\mathbb{R}^n$, then the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is onto.

Assume the columns of A span $\mathbb{R}^n$. Let $\bar{b}$ be any vector in $\mathbb{R}^n$. We have to find $\bar{x}$ satisfying $T(\bar{x}) = \bar{b}$. Let $\bar{a}_j$ denote the j^{th} column of A so that $A = [\bar{a}_1 | \bar{a}_2 | \cdots | \bar{a}_n]$. Since these columns span $\mathbb{R}^n$, there are scalars $c_1, c_2, \ldots, c_n$ satisfying

$$c_1\bar{a}_1 + c_2\bar{a}_2 + \cdots + c_n\bar{a}_n = \bar{b}.$$

So if we choose $\bar{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, then $A\bar{x} = \bar{b}$.

But this is the definition of T!

We have found $\bar{x} \in \mathbb{R}^n$ satisfying

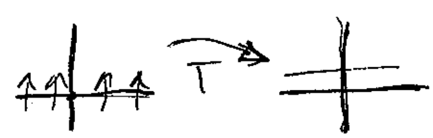
$$T(\bar{x}) := A\bar{x} = \bar{b}.$$

Therefore the function T is onto.

(b) [*3 points*] Give an example of a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is one-to-one and onto but is **not** a linear transformation. Explain.

We know that a linear transformation must send $\bar{0}$ to $\bar{0}$. So we can achieve this by any

* non-trivial translation
* rotation (by $\theta \neq 0°$) around an ~~axis~~ point other than the origin
* reflection across an axis not through the origin.



Example: $T(x,y) = (x, y+1)$

Clearly this is one-to-one since there is only one (x,y) pair getting shifted to any (a,b), namely $x=a$, $y=b-1$. Likewise it is onto since x,y always exist