C Term, Sections C01-C04 Instructor: W. J. Martin
Date: January 27, 2010

Matrices and Linear Algebra I – Test 1

NAME: SAMPLE SOLUTIONS Time: 110 minutes.

SECTION:
C01 (Erin, noon) C02 (Alyssa, 1pm) C03 (Grant, 2pm) C04 (Nghia, 3pm)

WRITE YOUR NAME IN THE SPACE ABOVE LABELED "NAME". CIRCLE YOUR SECTION.

Important Instructions: This test consists of 4 questions. Please answer all questions right on these pages, beginning each answer just below the statement of the question. If you need extra space (or space for rough work), use the back of the page, clearly indicating to which question the work pertains. **Show your work!** An incorrect answer with no explanation will receive no credit. A correct answer with no rough work will receive only partial credit. a

Please note that **no electronic devices** are permitted to be out in the open during the examination. Any use of electronic devices is considered cheating. Any communication between examinees during the examination is likewise an act of academic dishonesty.

DO NOT OPEN THIS TEST BOOKLET UNTIL SO INSTRUCTED.

This grid for graders' use only.

Question	1	2	3	4	Total (out of 40)
Score	10	10	10	10	40

1.) [*10 points*] Consider the following system of linear equations:

$$\begin{array}{rrrrrrcr} & & x_3 & -4x_4 & & +6x_6 & = & -9 \\ & & x_3 & -4x_4 & +x_5 & -x_6 & = & 1 \\ x_1 & -2x_2 & & +3x_4 & & +5x_6 & = & 8 \\ x_1 & -2x_2 & & +3x_4 & +x_5 & -2x_6 & = & 18 \end{array}$$

(a) Write down the augmented matrix corresponding to this system.

$$[A \,\vdots\, b] = \left[\begin{array}{cccccc:c} 0 & 0 & 1 & -4 & 0 & 6 & -9 \\ 0 & 0 & 1 & -4 & 1 & -1 & 1 \\ 1 & -2 & 0 & 3 & 0 & 5 & 8 \\ 1 & -2 & 0 & 3 & 1 & -2 & 18 \end{array}\right]$$

(b) Perform the row reduction algorithm on this matrix to obtain a matrix in **reduced row echelon form.**

$$[A \,\vdots\, b] \sim \begin{array}{c} \left[\begin{array}{cccccc:c} 1 & -2 & 0 & 3 & 0 & 5 & 8 \\ 0 & 0 & 1 & -4 & 0 & 6 & -9 \\ 0 & 0 & 1 & -4 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -7 & 10 \end{array}\right] \end{array} \begin{array}{l} R3 \\ R1 \\ R2 \\ R4 - R3 \end{array} \sim \left[\begin{array}{cccccc:c} 1 & -2 & 0 & 3 & 0 & 5 & 8 \\ 0 & 0 & 1 & -4 & 0 & 6 & -9 \\ 0 & 0 & 0 & 0 & 1 & -7 & 10 \\ 0 & 0 & 0 & 0 & 1 & -7 & 10 \end{array}\right] \begin{array}{l} \\ \\ R3 \\ -R2 \end{array}$$

$$\sim \left[\begin{array}{cccccc:c} 1 & -2 & 0 & 3 & 0 & 5 & 8 \\ 0 & 0 & 1 & -4 & 0 & 6 & -9 \\ 0 & 0 & 0 & 0 & 1 & -7 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \text{r.r.e.f.!} \\ \\ \\ R4 - R3 \end{array}$$

(c) Using part (b), find all solutions to the original linear system. Describe the solution set in parametric vector form.

$$\bar{X} = \begin{bmatrix} 8 \\ 0 \\ -9 \\ 0 \\ 10 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ -6 \\ 0 \\ 7 \\ 1 \end{bmatrix} \quad (r, s, t \text{ any real numbers})$$

By the way, here's the parametric form

$$\begin{cases} x_1 = 8 + 2r - 3s - 5t \\ x_2 = r \text{ (free)} \\ x_3 = -9 + 4s - 6t \\ x_4 = s \text{ (free)} \\ x_5 = 10 + 7t \\ x_6 = t \text{ (free)} \end{cases}$$

2.) For each of the following sets of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and target vectors $\mathbf{w}$, decide whether or not $\mathbf{w}$ is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. If "NO", explain. If "YES", then express $\mathbf{w}$ in terms of the members of S.

[NOTE: You are not absolutely required to use row reduction, but explain your answers.]

(a) *[3 points]* $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}$, $\quad \mathbf{w} = \begin{bmatrix} 9 \\ -9 \end{bmatrix}$.

We row reduce

$$[\bar{v}_1 | \bar{v}_2 | \cdots | \bar{v}_5 | \bar{w}] = \left[\begin{array}{ccccc|c} 0 & 2 & -1 & 4 & -4 & 9 \\ 0 & -1 & 2 & -2 & 2 & -9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 0 & 1 & -2 & 2 & -2 & 9 \\ 0 & 0 & 3 & 0 & 0 & -9 \end{array}\right] \begin{array}{l} -R2 \\ R1+2R2 \end{array} \sim \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & -3 \end{array}\right] \begin{array}{l} R1+\frac{2}{3}R2 \\ \frac{1}{3}R2 \end{array}$$

YES $\boxed{\bar{w} = 3\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$

(b) *[3 points]* $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix} \right\}$, $\quad \mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$.

Same strategy

$$[\bar{v}_1 | \bar{v}_2 | \bar{w}] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 3 & 6 & 6 \\ -2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 6 & 3 \\ 0 & 4 & 2 \end{array}\right] \begin{array}{l} R1 \\ R2-3R1 \\ R3+2R1 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \frac{1}{6}R2 \\ R3-\frac{2}{3}R2 \end{array}$$

YES $\boxed{\bar{w} = 1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 6 \\ 4 \end{bmatrix}}$

$\bar{w} = \bar{v}_1 + \frac{1}{2}\bar{v}_2$

(c) *[4 points]* $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} \right\}$, $\quad \mathbf{w} = \begin{bmatrix} 5 \\ 3 \\ 15 \\ 20 \end{bmatrix}$.

Same Strategy again

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 2 & 3 \\ 3 & 3 & 3 & 15 \\ 4 & 4 & 5 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R1 \\ R2-2R1 \\ R3-3R1 \\ R4-4R1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

unique solution
$c_3 = 0$
$c_2 = -7$
$c_1 + c_2 + c_3 = 5$

YES

$\boxed{\bar{w} = 12\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - 7\begin{bmatrix} 1 \\ 3 \\ 3 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}}$

3.) For each of the following square matrices A and eigenvalues λ, find all eigenvectors of A corresponding to eigenvalue λ. (As always, show your work.)

(a) [*3 points*] $A = \begin{bmatrix} 5 & -3 \\ 2 & -2 \end{bmatrix}$, $\lambda = 4$

$$A - \lambda I = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$$

$$[A - \lambda I \,\vdots\, 0] = \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 2 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R2 - 2R1$$

Solution set

$$\begin{cases} x_1 = 3r \\ x_2 = r \quad (r \in \mathbb{R}) \end{cases}$$

Eigenvectors

$$\left\{ r\begin{bmatrix} 3 \\ 1 \end{bmatrix} : r \neq 0 \right\}$$

(b) [*3 points*] $A = \begin{bmatrix} -3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\lambda = -3$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$[A - \lambda I \,\vdots\, 0] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -9 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution set

$$\begin{cases} x_1 = 9r \\ x_2 = -3r \\ x_3 = r \text{ (free)} \end{cases}$$

Eigenvectors

$$\left\{ r\begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix} : r \neq 0 \right\}$$

(c) [*4 points*] $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & -1 & \\ & & & 0 \end{bmatrix}$$

$$[A - \lambda I \,\vdots\, 0] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad -R3$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ \text{free} & \text{free} & & \text{free} \end{array}$$

Solution set

$$\begin{cases} x_1 = r \\ x_2 = s \\ x_3 = 0 \quad (r,s,t \in \mathbb{R}) \\ x_4 = t \end{cases}$$

Eigenvectors

$$\left\{ \begin{bmatrix} r \\ s \\ 0 \\ t \end{bmatrix} : \begin{array}{l} r,s,t \text{ any} \\ \text{numbers} \\ \underline{\text{except}} \text{ all zero} \end{array} \right\}$$

4(a) [*5 points*] Suppose A is an 8×5 matrix whose columns are linearly independent. If $\mathbf{p} = (1, 2, 3, 0, 0)$ is a solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (5, 0, 4, -3, 1, 1, 1, 1)$, then find **all** solution to $A\mathbf{x} = \mathbf{b}$. Explain.

$\boxed{\text{The full solution set is just } \{p\}.}$

Since the columns of A are linearly independent, the homogeneous system $A\bar{x} = \bar{0}$ has only the trivial solution (by some theorem). But the solutions to $A\bar{x} = \bar{b}$ are all given by

$$\bar{v} = \bar{p} + \bar{v}_h$$

where $\bar{v}_h$ is a solution to $A\bar{x} = \bar{0}$. Since there is only one choice for $\bar{v}_h$ (namely $\bar{v}_h = \bar{0}$), there is only one choice for $\bar{v}$: $\bar{v} = \bar{p}$.

(b) [*5 points*] Give an example in $\mathbb{R}^4$ of a set of vectors which is linearly independent yet does not span $\mathbb{R}^4$. Explain.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Any non-zero vector gives us a linearly independent set. Yet this set cannot span $\mathbb{R}^4$. For example

$\bar{w} = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the span of S since it is not a scalar multiple of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.