Matrices & Linear Algebra I
C Term, Sections C01-C04
March 3, 2010

Remedial Worksheet: Finding Roots of Polynomials

Factoring Quadratics

We all recall the quadratic formula, but things are much simpler if you suspect, ahead of time, that the roots will all be integers.

If $f(\lambda) = \lambda^2 + b\lambda + c$ factors as $(\lambda - r)(\lambda - s)$, then $r + s = -b$ and $rs = c$.

Find the roots of each polynomial:

1. $\lambda^2 + 5\lambda + 4$
2. $\lambda^2 - 5\lambda + 4$
3. $\lambda^2 + 3\lambda - 4$
4. $\lambda^2 - 3\lambda - 4$
5. $\lambda^2 - 5\lambda - 6$
6. $\lambda^2 - 6\lambda + 9$
7. $\lambda^2 - 7\lambda + 12$
8. $\lambda^2 - 5\lambda - 36$
9. $\lambda^2 + 5\lambda - 36$
10. $\lambda^2 + 5\lambda - 14$
11. $\lambda^2 - 5\lambda - 14$
12. $\lambda^2 - 9\lambda + 14$

Nicely Behaved Cubics

Find the roots of each polynomial:

13. $\lambda^3 + 5\lambda^2 + 4\lambda$
14. $\lambda^3 + 15\lambda^2 + 44\lambda$
15. $\lambda^3 - 5\lambda^2 + 6\lambda$
16. $\lambda^3 - 5\lambda^2 - 6\lambda$
17. $\lambda^3 + \lambda^2 - 2\lambda$
18. $\lambda^3 + 2\lambda^2 + \lambda$
19. $\lambda^3 + 9\lambda^2 + 14\lambda$
20. $\lambda^3 + 8\lambda^2 - 20\lambda$

One and Minus One as Roots

A polynomial $f(\lambda)$ has $\lambda = 1$ as a root if $f(1) = 0$, right? But $f(1)$ is just the sum of all the coefficients of f (just ignore all the λs). So it's easy to check if one is a root. Likewise, if

$$f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_2\lambda^2 + a_1\lambda + a_0$$

then

$$f(-1) = a_0 - a_1 + a_2 - \cdots + (-1)^{n-1}a_{n-1} + (-1)^n.$$

Once we find that $\lambda = 1$ is a root, we have to divide the polynomial by $(\lambda - 1)$ and factor the rest. Likewise, if we find that $\lambda = -1$ is a root, we divide the polynomial by $(\lambda + 1)$ and continue factoring.

Always keep in mind that the coefficients of the polynomial give you hints about the roots: the constant term is the product of the roots (up to sign) and the coefficient of the second highest term is the negative of the sum of the roots.

Find the roots of each polynomial:

21. $\lambda^3 + 3\lambda^2 + 3\lambda + 1$
22. $\lambda^3 - 3\lambda^2 + 3\lambda - 1$
23. $\lambda^3 - \lambda^2 - \lambda + 1$
24. $\lambda^3 + \lambda^2 - \lambda - 1$
25. $\lambda^3 - 7\lambda + 6$
26. $\lambda^3 - 2\lambda^2 - 5\lambda + 6$
27. $\lambda^3 - \lambda^2 - 9\lambda + 9$
28. $\lambda^3 - 6\lambda^2 - 9\lambda + 14$
29. $\lambda^3 - 10\lambda^2 + 23\lambda - 14$
30. $\lambda^3 + 4\lambda^2 - 19\lambda + 14$
31. $\lambda^3 + 8\lambda^2 + 5\lambda - 14$

More Test-Like Examples

The other thing we all need to keep in mind is that polynomials (like mathematical structure, in general) do not come to us in such clean form. Before we can apply the above skills, we must carefully multiply certain expressions out, re-arrange things, and collect like terms.

Find the roots of each polynomial:

32. $\lambda(\lambda - 1)(\lambda - 2) + 2(\lambda - 1)(\lambda - 2) - 5(\lambda - 1)$
33. $(\lambda + 1)^3 + 5\lambda(\lambda - 4) - 3\lambda - 1$
34. $\lambda^2(\lambda - 10) + 23(\lambda - 1) + 9$
35. $(\lambda + 1)^3 - \lambda(\lambda + 2) - 1$

36. $\lambda(\lambda+2)(\lambda+6)-7(\lambda+2)+9-9$

37. $(\lambda-1)^3+2(\lambda-1)^2$

38. $\lambda(\lambda+1)(\lambda-1)+6(1-\lambda)$

39. $(-1-\lambda)(-2-\lambda)(2-\lambda)-6\lambda(2-\lambda)+2\lambda-4$

40. $(\lambda-1)^3+6(\lambda-2)^2+24\lambda-22$

41. $(-1-\lambda)(-2-\lambda)(-6-\lambda)+8(-1-\lambda)-16(-2-\lambda)+2(-6-\lambda)$

Hint: Each of these, when properly expanded, is essentially a polynomial you've already seen above.

Answers

1. $(\lambda+4)(\lambda+1)$
2. $(\lambda-1)(\lambda-4)$
3. $(\lambda+4)(\lambda-1)$
4. $(\lambda+1)(\lambda-4)$
5. $(\lambda+1)(\lambda-6)$
6. $(\lambda-3)^2$
7. $(\lambda-3)(\lambda-4)$
8. $(\lambda+4)(\lambda-9)$
9. $(\lambda+9)(\lambda-4)$
10. $(\lambda+7)(\lambda-2)$
11. $(\lambda+2)(\lambda-7)$
12. $(\lambda-2)(\lambda-7)$
13. $\lambda(\lambda+4)(\lambda+1)$
14. $\lambda(\lambda+11)(\lambda+4)$
15. $\lambda(\lambda-2)(\lambda-3)$
16. $\lambda(\lambda+1)(\lambda-6)$
17. $\lambda(\lambda+2)(\lambda-1)$
18. $\lambda(\lambda+1)^2$
19. $\lambda(\lambda+7)(\lambda+2)$
20. $\lambda(\lambda+10)(\lambda-2)$
21. $(\lambda+1)^3$
22. $(\lambda-1)^3$
23. $(\lambda+1)(\lambda-1)^2$
24. $(\lambda-1)(\lambda+1)^2$
25. $(\lambda-1)(\lambda-2)(\lambda+3)$
26. $(\lambda-1)(\lambda-3)(\lambda+2)$
27. $(\lambda-1)(\lambda+3)(\lambda-3)$
28. $(\lambda-1)(\lambda-7)(\lambda+2)$
29. $(\lambda-1)(\lambda-2)(\lambda-7)$
30. $(\lambda-1)(\lambda-2)(\lambda+7)$
31. $(\lambda-1)(\lambda+7)(\lambda+2)$
32. $(\lambda-1)(\lambda+3)(\lambda-3)$
33. $\lambda(\lambda+10)(\lambda-2)$
34. $(\lambda-1)(\lambda-2)(\lambda-7)$
35. $\lambda(\lambda+1)^2$
36. $(\lambda-1)(\lambda+7)(\lambda+2)$
37. $(\lambda+1)(\lambda-1)^2$
38. $(\lambda-1)(\lambda-2)(\lambda+3)$
39. $-\lambda(\lambda-2)(\lambda-3)$
40. $(\lambda+1)^3$
41. $-\lambda(\lambda+7)(\lambda+2)$