Linear Algebra
C Term, Sections C01-C04
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# Linear Algebra Quiz 7

## Sample Solutions

For each of the following matrices, find a basis for $\mathbb{R}^n$ consisting entirely of eigenvectors for the matrix $A$.

**(a)** $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, eigenvalues 2, 5.

Solution: We don't even need to row reduce: we see that

$$A\mathbf{e}_1 = 2\mathbf{e}_1, \qquad A\mathbf{e}_2 = 2\mathbf{e}_2, \qquad A\mathbf{e}_3 = 5\mathbf{e}_3$$

so that the standard basis $\mathcal{S} = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ is a basis for $\mathbb{R}^3$ where each basis vector is an eigenvector for $A$.

**(b)** $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, eigenvalues 2, 3.

Solution: For $\lambda_1 = 3$, we row reduce $A - \lambda_1 I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$ to find eigenvector $\mathbf{v}_1 = (1,2)$. Likewise, we row reduce $A - 2I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ to find eigenvector $\mathbf{v}_1 = (1,1)$. So $\mathcal{B} = \{(1,2),\ (1,1)\}$ is a basis meeting the required condition.

**(c)** $A = \begin{bmatrix} 10 & -9 & 6 \\ 4 & -2 & 4 \\ 2 & -3 & 6 \end{bmatrix}$, eigenvalues 4, 6.

Solution: For $\lambda_1 = 6$, we row reduce

$$A - 6I = \begin{bmatrix} 4 & -9 & 6 \\ 4 & -8 & 4 \\ 2 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

to find eigenvector $\mathbf{v}_1 = (3,2,1)$. Next, we set $\lambda_2 = 4$ and row reduce

$$A - \lambda_2 I = \begin{bmatrix} 6 & -9 & 6 \\ 4 & -6 & 4 \\ 2 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

to find two linearly independent eigenvectors $\mathbf{v}_2 = (3, 2, 0)$ and $\mathbf{v}_3 = (-1, 0, 1)$.

Putting these bases together, we find a basis for $\mathbb{R}^3$ consisting entirely of eigenvectors for $A$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

**(d)** $A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, eigenvalues 2, 1 and 3.

Solution: Let's set $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = \lambda_4 = 1$. Starting with $\lambda_1$, we find

$$A - 3I = \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now the solutions to the homogeneous system are clear: we get eigenvector $\mathbf{v}_1 = (-2, -1, 2, 0)$. Next, we do the same for $\lambda_2 = 2$:

$$A - 2I = \begin{bmatrix} 0 & 0 & -1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us $\mathbf{v}_2 = \mathbf{e}_1$ as the corresponding eigenvector.

The last eigenvalue is $\theta = 1$ with multiplicity two. We row reduce

$$A - I = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us only <u>one</u> eigenvector, $\mathbf{v}_3 = \mathbf{e}_2$, for $\theta = 1$.

So we have a <u>defective</u> eigenvalue: $\theta = 1$ has algebraic multiplicity two yet geometric multiplicity only one. This means that the matrix $A$ is <u>not</u> diagonalizable (due to the professor's typo). So no such basis of eigenvectors exists.