Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Quiz 6
SAMPLE SOLUTIONS

Determine the kernel of the given linear transformation. In each case, describe the kernel in parametric vector form.

(a) $T : \mathbb{R}^4 \to \mathbb{R}^2$ via $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

SOLUTION: In the case where $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A, the kernel is just the null space of the matrix. So we row reduce

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and find

$$\operatorname{Ker} T = \left\{ r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \ : \ r, s \in \mathbb{R} \right\}$$

(b) $T : \mathbb{P}_3 \to \mathbb{P}_3$ via $T : \mathbf{p}(t) \mapsto \mathbf{p}(t) - t\mathbf{p}'(t)$

SOLUTION: We first need to find a simple computational description for T. We suppose we know the coefficients of polynomial $\mathbf{p}(t)$ and compute

$$\begin{aligned} \mathbf{p}(t) &= a + bt + ct^2 + dt^3 \\ \mathbf{p}'(t) &= b + 2ct + 3dt^2 \\ t\mathbf{p}'(t) &= bt + 2ct^2 + 3dt^3 \\ \mathbf{p}(t) - t\mathbf{p}'(t) &= a - ct^2 - 2dt^3 \ . \end{aligned}$$

The kernel of T consists of all polynomials $\mathbf{p}(t)$ for which $T(\mathbf{p}(t))$ is the zero polynomial. So we need to find all solutions to

$$(\mathbf{p}(t) - t\mathbf{p}'(t) =) \quad a - ct^2 - 2dt^3 = \mathbf{0} \quad \left(= 0 + 0t + 0t^2 + 0t^3\right).$$

Now it's easy: we need $a = c = d = 0$ and b is a free parameter. So

$$\operatorname{Ker} T = \left\{ 0 + bt + 0t^2 + 0t^3 \ \mid \ b \in \mathbb{R} \ \right\}.$$

In parametric vector form, we can can use the basis $\{t\}$ and write

$$\operatorname{Ker} T = \{bt \mid b \in \mathbb{R}\}.$$

(c) $T : M_{3\times 3} \longrightarrow M_{3\times 3}$ via $T : A \mapsto (A + A^\top)$

SOLUTION: Here, we can be a bit smarter. We see that $T(A) = A + A^\top$. So a matrix A is in the kernel of T if and only if $A + A^\top = 0$, the 3×3 matrix of all zeros. So A is in Ker$T$ precisely when $A^\top = -A$. Thus, the kernel of T is exactly the set of all skew-symmetric matrices:

$$\operatorname{Ker} T = \left\{ r \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + s \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \middle| \, r, s, t \in \mathbb{R} \right\}.$$

(d) $T : \mathbb{R}^2 \longrightarrow M_{2\times 3}$ via $T(x, y) = \begin{bmatrix} 0 & x - 2y & 2y - x \\ 2x - 4y & 4y - 2x & 0 \end{bmatrix}$

SOLUTION: Again, we need to find all of the vectors that map to the zero vector, which in this case is the 2×3 matrix of all zeros. We solve

$$\begin{bmatrix} 0 & x - 2y & 2y - x \\ 2x - 4y & 4y - 2x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for x and y by setting all entries of $T(x, y)$ to zero:

$$\begin{aligned} 0 &= 0 \\ x - 2y &= 0 \\ -x + 2y &= 0 \\ 2x - 4y &= 0 \\ -2x + 4y &= 0 \\ 0 &= 0 \end{aligned}$$

The solution set is then $\operatorname{Ker} T = \left\{ r \begin{bmatrix} 2 \\ 1 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$.