Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Quiz 3

Sample Solutions

For each of the following functions from $\mathbb{R}^3$ to $\mathbb{R}^2$, decide whether or not T is a linear transformation. If you conclude that T *is* a linear transformation, then give the images of $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$, under T. If you conclude that T is *not* a linear transformation, then give explicit values where the function fails to satisfy the definition of a linear transformation.

1.) $T(x,y,z) = (2x+y,\ 0)$

Solution: Since each term in the function's output is a linear expression (with zero constant term), this is a linear transformation. We have

$$T(\mathbf{e}_1) = (2,0), \qquad T(\mathbf{e}_2) = (1,0), \qquad T(\mathbf{e}_3) = (0,0).$$

2.) $T(x,y,z) = (y-10z,\ \sin x)$

Solution: The first component of the output looks fine, but there is no way $\sin x$ can be linear. (Right??) Let's make this clear. Take $\mathbf{u} = (\frac{\pi}{2},0,0)$ and $\mathbf{v} = (\frac{\pi}{2},0,0)$. Then $\mathbf{u}+\mathbf{v} = (\pi,0,0)$. So

$$T(\mathbf{u}+\mathbf{v}) = T(\pi,0,0) = (0,0)$$

while

$$\begin{aligned} T(\mathbf{u}) + T(\mathbf{v}) &= T(\frac{\pi}{2},0,0) + T(\frac{\pi}{2},0,0) \\ &= (1,0) + (1,0) \\ &= (2,0) \end{aligned}$$

since $\sin\frac{\pi}{2} = 1$ while $\sin\pi = 0$. So $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ and the definition fails: T is **not** a linear transformation.

3.) $T(x,y,z) = (yz,\ 0)$

Solution: This time, the first component is the suspicious one. Take $\mathbf{u} = (0,1,0)$ and $\mathbf{v} = (0,0,1)$. Then $\mathbf{u}+\mathbf{v} = (0,1,1)$. So

$$T(\mathbf{u}+\mathbf{v}) = T(0,1,1) = (1,\ 0)$$

while

$$\begin{aligned} T(\mathbf{u}) + T(\mathbf{v}) &= T(0,1,0) + T(0,0,1) \\ &= (0,0) + (0,0) \\ &= (0,0) \end{aligned}$$

So $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ and the definition fails: T is **not** a linear transformation.

4.) $T(x, y, z) = (4x - 3y + z, x + 9z)$

SOLUTION: Since each term in the function's output is a linear expression (with zero constant term), this is a linear tranformation. We have

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix},$$

$$T(\mathbf{e}_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 9 \end{bmatrix}.$$

(Note here that writing the vectors in their ordinary form – as 3×1 matrices – takes up more space than the abbreviations such as $\mathbf{e}_1 = (1, 0, 0)$ that we used in earlier problems.)