Linear Algebra
C Term, Sections C01-C04
W. J. Martin
January 25, 2010

Linear Algebra Quiz 2

SAMPLE SOLUTIONS

For each of the following sets of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and target vector $\mathbf{b}$, determine whether or not $\mathbf{b}$ is in the span of S. If so, express $\mathbf{b}$ in terms of the members of S.

(a) In $\mathbb{R}^2$, $\quad S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$

SOLUTION: As indicated in the statement of the problem, we give the four vectors in S the names $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, respectively. Since $4\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $4\mathbf{v}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, we easily spot a solution. Yes, $\mathbf{b}$ belongs to Span S: $\mathbf{b} = -4\mathbf{v}_1 + 4\mathbf{v}_2$.

(b) In $\mathbb{R}^2$, $\quad S = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -6 \\ 10 \end{bmatrix}, \begin{bmatrix} 15 \\ -25 \end{bmatrix} \right\}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$

SOLUTION: The equivalent matrix equation is $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 15 \\ -5 & 10 & -25 \end{bmatrix}$$

So we row-reduce the augmented matrix

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 3 & -6 & 15 & 1 \\ -5 & 10 & -25 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 1/3 \\ 0 & 0 & 0 & 17/3 \end{array}\right]$$

We can stop here; we've reached echelon form with a pivot in the last column. So the linear system is INCONSISTENT. Answer: NO, $\mathbf{b}$ is not in the span of S.

(c) In $\mathbb{R}^3$, $\quad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$

SOLUTION: The equivalent matrix equation is $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

So we row-reduce the augmented matrix

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

Again, we've reached echelon form with a pivot in the last column. So the linear system is INCONSISTENT. Answer: NO, $\mathbf{b}$ is not in the span of S.

(d) In $\mathbb{R}^3$, $\quad S = \left\{ \begin{bmatrix} 9/4 \\ -5/2 \\ 1/8 \end{bmatrix}, \begin{bmatrix} -29/13 \\ 5/16 \\ -1/17 \end{bmatrix}, \begin{bmatrix} 14/5 \\ 12/7 \\ -2/3 \end{bmatrix} \right\}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

SOLUTION: This one is trivial. If we give the three vectors in S the names $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, respectively, we have

$$\mathbf{b} = \mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3.$$

So YES, the vector $\mathbf{b}$ DOES belong to the span of S.