Row Reduction

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Overview of Today's Class

Today, we will discuss

- elementary row operations
- the Row Reduction Algorithm
- row echelon form (REF)
- reduced row echelon form (RREF)
- inconsistent vs. consistent linear systems
- unique solution vs. infinitely many solutions
- efficiently describing the solution set to a linear system

Example 1

Consider the linear system

$$\begin{array}{rcrcr} 3\ x_1 & + & 2\ x_2 & = & 22 \\ x_1 & - & 3\ x_2 & = & -11 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rr|r} 3 & 2 & 22 \\ 1 & -3 & -11 \end{array}\right]$$

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then use Row 1 to eliminate the nonzero entry in position (2,1):

$$\sim \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 11 & 55 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - 3(R1) \end{array}$$

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We now have a matrix in **row echelon form**.

Row Echelon Form (REF)

$$\left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 11 & 55 \end{array}\right]$$

The first nonzero entry in each row will be called a **pivot** and its location is a **pivot position**.

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- each pivot occurs to the *right* of all pivots above it
- all entries directly below a pivot position must be zero

This is the definition of **row echelon form** (REF).

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But we prefer to proceed further in our row reduction algorithm.

$$\left[\begin{array}{rr|r} 3 & 2 & 22 \\ 1 & -3 & -11 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -3 & -11 \\ 0 & 11 & 55 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -3 & -11 \\ 0 & 1 & 5 \end{array}\right] \begin{array}{l} (R1) \\ \frac{1}{11}(R2) \end{array}$$

$$\sim \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array}\right] \begin{array}{l} (R1) + 3(R2) \\ (R2) \end{array}$$

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We have now arrived at **reduced row echelon form** (RREF).

Reduced Row Echelon Form (RREF)

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 5 \end{array}\right]$$

We have

- rows consisting entirely of zeros, if any, appear at the bottom
- each pivot occurs to the *right* of all pivots above it
- all pivots are equal to **one**
- all entries directly **above and** below a pivot position must be zero

Reading off a Solution

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Since all of these operations on rows are **reversible** (think about it) the systems represented by any two of these matrices are **equivalent**: they have the same solution set.

So the *original system*

$$\begin{array}{rrrrrrr} 3 & x_1 & + & 2 & x_2 & = & 22 \\ & x_1 & - & 3 & x_2 & = & -11 \end{array}$$

has the same solutions as the reduced system

$$\begin{array}{rrrl} x_1 & & = & 4 \\ & x_2 & = & 5 \end{array}$$

and we know what these are.

Elementary Row Operations

The row reduction algorithm is a sequence of carefully executed steps.

The only valid steps for us are the following (p7):

- (e1) Add a multiple of one row to another (one step at a time)
- (e2) Swap any two rows
- (e3) Multiply any row by any nonzero constant

Now for the Theory

We next present two very important basic theorems.

We will use both of these many times in the course.

Theorem 1: The RREF is Unique

Given any $m \times n$ matrix A, there is exactly one matrix in reduced row echelon form which is row equivalent to A.

So:
Two $m \times n$ matrices A and B are row equivalent if and only if they have the same RREF.

Theorem 2: Sizes of Solution Sets

Suppose we have a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n variables.

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- If there is a pivot in the last column (corr. to $\mathbf{b}$), then the system is **inconsistent**: it has **no solution**:

$$[0\ 0\ 0\ \cdots\ 0 \mid \bar{b}_i]$$

- If all pivots occur to the left of the vertical dashed line, then the system is **consistent**
- If every column to the left of the vertical dashed line (i.e., each of the n columns corr. to variables) has a pivot, then there is a **unique solution** to the linear system
- If the system is consistent and there are less than n pivots, then there are **infinitely many solutions**.

Mnemonic Form: Sizes of Solution Sets

Suppose we have a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n variables.
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Suppose we have a linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n variables.
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- **Pivot in last column:** no solution
- **Pivot in every column except last:** unique solution
- **No pivot in last column, but also no pivot in some other column:** free variable, infinitely many solutions

Example 2

Consider the linear system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & x_3 & & & = & 5 \\ & & x_2 & & & + & x_4 & = & 1 \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & 4 \end{array}$$

with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & 4 \end{array}\right]$$

(Note that if we replaced the 4 by anything else, we would get an inconsistent system.)

Example 2 Row Reduction

We row reduce the augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3) - (R1) \end{array}$$

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$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3)+(R2) \end{array}$$

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This matrix is in reduced row echelon form. The variables x_1 and x_2 are basic and the variables x_3 and x_4 are free. The solution set is now easy to write down:

$$\begin{aligned} x_1 &= 5 - r \\ x_2 &= 1 \qquad - s \\ x_3 &= \quad r \\ x_4 &= \qquad s \end{aligned}$$

Example 3

Our last linear system is

$$
\begin{array}{rrrrrrrrrrr}
 & & & & & & x_3 & + & x_4 & = & 4 \\
 & x_1 & - & 2 & x_2 & & & - & x_4 & = & 1 \\
2 & x_1 & - & 4 & x_2 & & & + & x_4 & = & -1 \\
2 & x_1 & - & 4 & x_2 & + & 2 & x_3 & & = & 10
\end{array}
$$

with augmented matrix

$$
\left[\begin{array}{rrrr|r}
0 & 0 & 1 & 1 & 4 \\
1 & -2 & 0 & -1 & 1 \\
2 & -4 & 0 & 1 & -1 \\
2 & -4 & 2 & 0 & 10
\end{array}\right]
$$

This is the first spot where we need to swap rows in order to get a nonzero pivot.

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We row reduce the augmented matrix

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$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 2 & 2 & 8 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3) - 2(R1) \\ (R4) - 2(R1) \end{array}$$

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Example 3 Row Reduction Continued

We began with the augmented matrix

$$\left[\begin{array}{rrrr|r} 0 & 0 & 1 & 1 & 4 \\ 1 & -2 & 0 & -1 & 1 \\ 2 & -4 & 0 & 1 & -1 \\ 2 & -4 & 2 & 0 & 10 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{from above}$$

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$$\sim \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) + \frac{1}{3}(R3) \\ (R2) - \frac{1}{3}(R3) \\ \frac{1}{3}(R3) \\ (R4) \end{array}$$

This matrix is in reduced row echelon form. The variable x_2 is free. So we easily find all solutions:

$$\begin{aligned} x_1 &= 2r \\ x_2 &= r \\ x_3 &= 5 \\ x_4 &= -1 \end{aligned}$$