THREE PROBLEMS – SAME SOLUTION

* Resource allocation in manufacturing
* Robot motion planning
* Inverting a geometric transformation

RESOURCE ALLOCATION

A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes unfinished slabs of steel as input (1 ton each) and can produce either of two semi-finished products: bands and coils.

The mills two products come off the rolling line at different rates: Bands at $200 \frac{\text{tons}}{\text{hr}}$, Coils at $140 \frac{\text{tons}}{\text{hr}}$

They also produce different profits:
\$25/ton for Bands, \$30/ton for Coils

Q: What combination of bands and coils uses 40 hours of mill time and yields \$180,000 profit?

I. RESOURCE ALLOCATION

Our company,
produces three products
GOODNESS, HOPE & CHARITY
from three raw materials
SNOT, SPIT & FILTH

* Each ounce of GOODNESS requires
  - one ton of SNOT
  - one ton of SPIT
* Each ounce of HOPE produced requires
  - one ton of SNOT
  - two tons of SPIT
  - one ton of FILTH
* Each ounce of CHARITY is made of
  - one ton of SNOT
  - two tons of FILTH

Question: What combination of the above products utilizes exactly
- 9 tons of SNOT
- 8 tons of SPIT

and
- 11 tons of FILTH ?

SOLVING A SYSTEM OF LINEAR EQUATIONS

If we let

x_1 = # ounces of GOODNESS to be produced

x_2 = # ounces of HOPE to be produced

x_3 = # ounces of CHARITY to be produced

We get

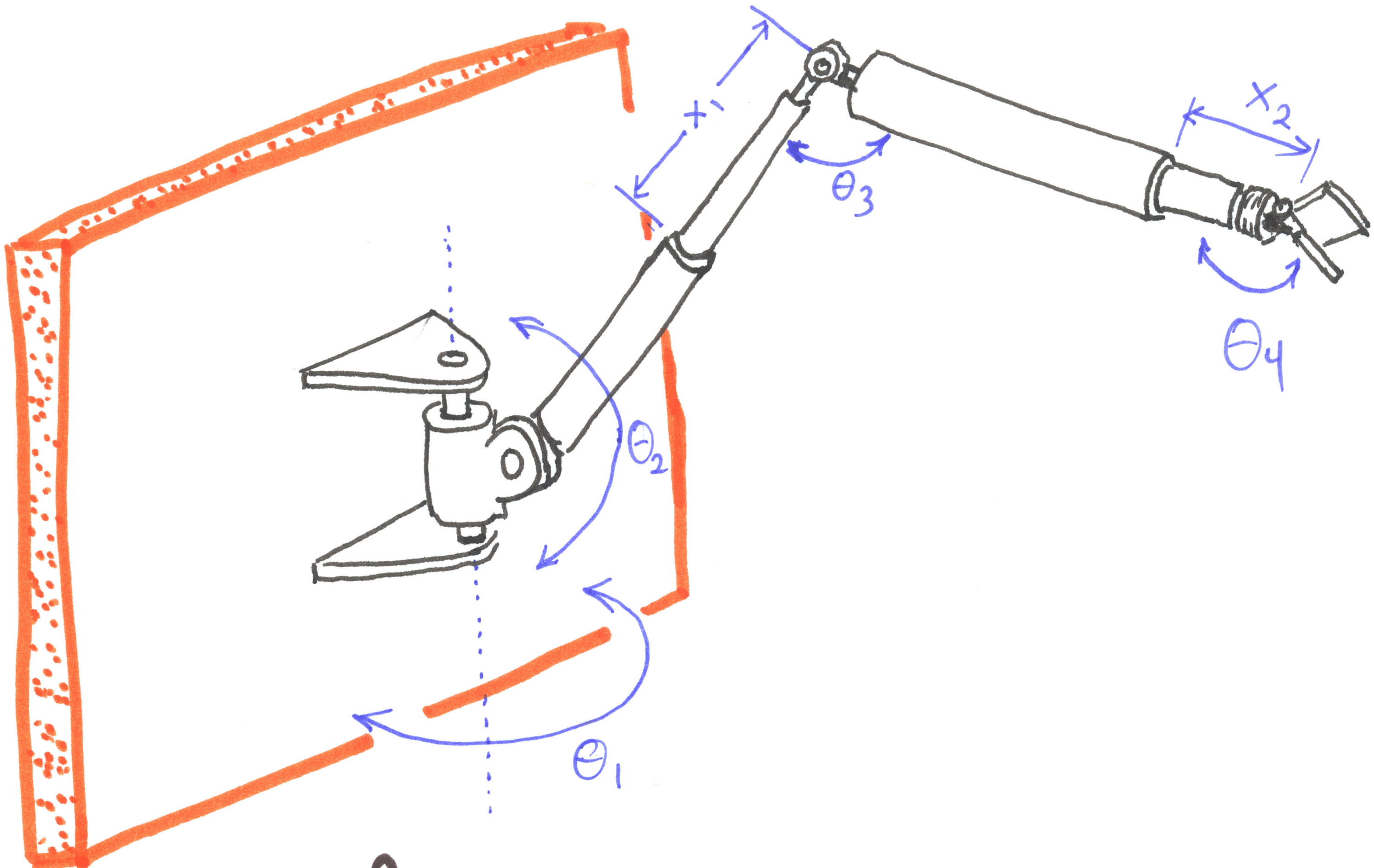
$$x_1 + x_2 + x_3 = 9 \quad \text{(total SNOT used, in tons)}$$

$$x_1 + 2x_2 = 8 \quad \text{(" SPIT " ")}$$

$$x_2 + 2x_3 = 11 \quad \text{(" FILTH " ")}$$

We will solve this linear system on the blackboard.

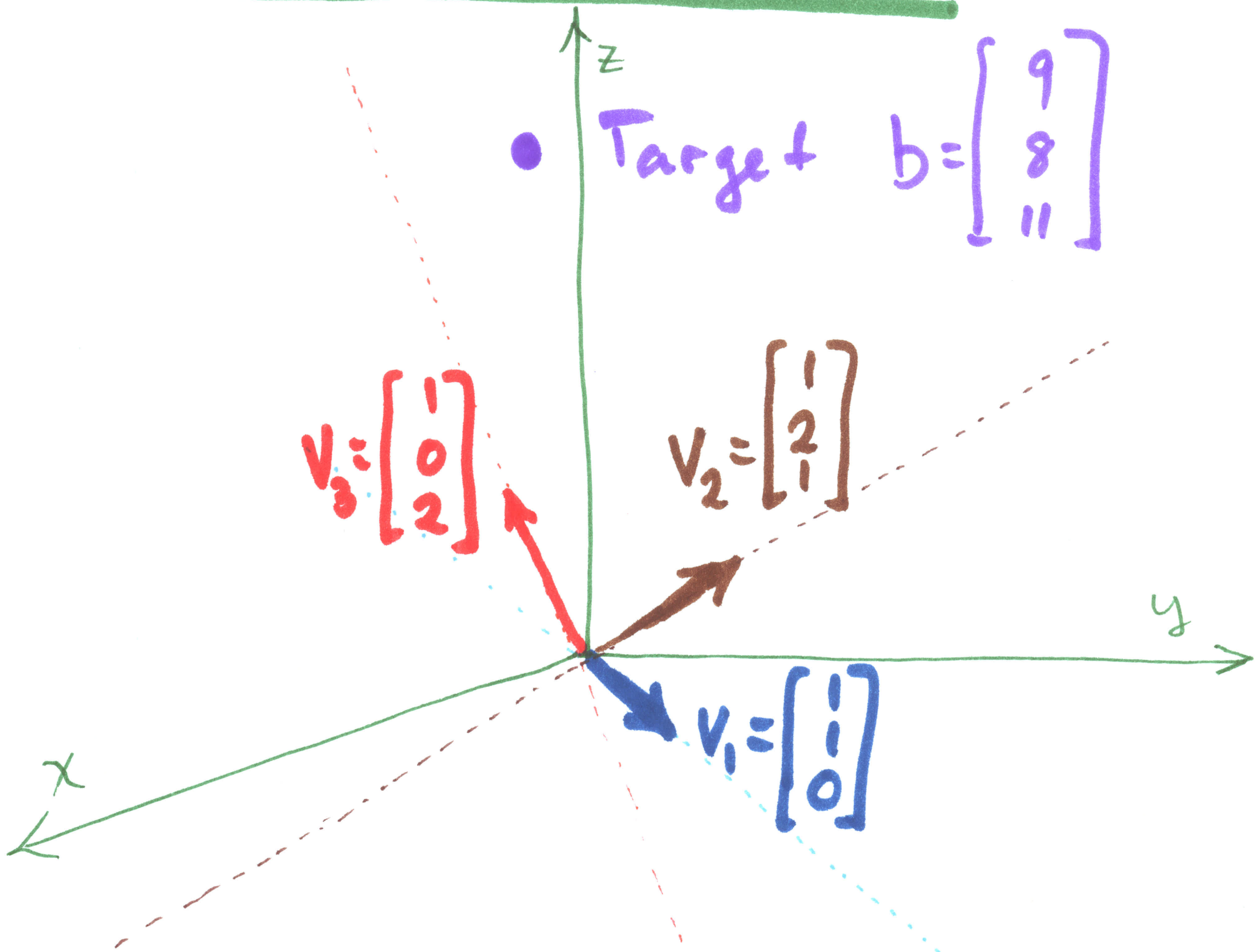
II. Robot Motion Planning



A function mapping 6-dimensional joint space to 3-dimensional real space

$$T : [0,\pi]^3 \times [0,2\pi] \times [0,5]^2 \longrightarrow \mathbb{R}^3$$

Robot Motion Planning



Suppose our robotic arm must first move some distance, x_1, in direction v_1

then some distance x_2 in direction v_2

and then some amount x_3 in direction v_3

How can we get the end-effector to the target?

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Vector Equations

Our problem can be phrased as: find real numbers x_1, x_2, x_3 with

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = b$$

i.e.,

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 11 \end{bmatrix}$$

Vector Algebra:

Scalar multiplication: multiply every entry of the vector by a constant

Vector addition: add corresponding entries

Vector equation: all corresponding entries must be equal

We get

$$\begin{aligned} x_1 + x_2 + x_3 &= 9 \\ x_1 + 2x_2 &= 8 \\ x_2 + 2x_3 &= 11 \end{aligned}$$

III. INVERTING A LINEAR TRANSFORMATION

Consider real 3-space

$$\mathbb{R}^3 = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

where $\mathbb{R}$ denotes the real numbers, and a transformation

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

which sends x to

$$T(x) = Ax$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ is a } 3 \times 3 \text{ matrix}$$

and

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 \\ x_2 + 2x_3 \end{bmatrix}$$

Question: Is there a point x that gets transformed to $b = \begin{bmatrix} 9 \\ 8 \\ 11 \end{bmatrix}$?

I.e., can we solve $Ax = b$?

Solution Method: Row reduce augmented matrix

$$[A \,\vdots\, b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 0 & 8 \\ 0 & 1 & 2 & 11 \end{array}\right]$$