Linear Algebra
C Term, Sections C01-C04
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Examples: Change-of-Coordinate Matrices

Here, we collect a few examples of change-of-coordinates matrices.

Let V be a vector space of dimension n and let $\mathcal{B}$ and $\mathcal{C}$ be two bases for V. Then the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is defined by the equation

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \; [\mathbf{x}]_{\mathcal{B}} \;=\; [\mathbf{x}]_{\mathcal{C}} \; .$$

That is, it is the unique matrix that transforms any coordinate vector relative to basis $\mathcal{B}$ into the coordinate vector of the same element $\mathbf{x}$ relative to basis $\mathcal{C}$. It is not too hard to prove that, if $\mathcal{B} \;=\; \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, then

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \; [[\mathbf{b}_1]_{\mathcal{C}} \; |[\mathbf{b}_2]_{\mathcal{C}} \; | \cdots | \; [\mathbf{b}_n]_{\mathcal{C}}] \, .$$

This gets really simple when $\mathcal{B} = \mathcal{C}$: for any basis $\mathcal{B}$, we see that $\underset{\mathcal{B}\leftarrow\mathcal{B}}{P} = \; I$.

Example 1: Let's start with $V \;=\; \mathbb{R}^2$, with standard basis $\mathcal{S} \;=\; \{\mathbf{e}_1 \;=\; (1,0),\; \mathbf{e}_2 \;=\; (0,1)\}$ and second basis

$$\mathcal{B} \;=\; \left\{ \mathbf{v}_1 \;=\; \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \; \mathbf{v}_2 \;=\; \begin{bmatrix} 2 \\ -7 \end{bmatrix} \right\}$$

Since $\mathbf{v}_1 \;=\; \mathbf{e}_1 - 4\mathbf{e}_2$ and $\mathbf{v}_2 \;=\; 2\mathbf{e}_1 - 7\mathbf{e}_2$, we have

$$[\mathbf{v}_1]_{\mathcal{S}} \;=\; \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \qquad [\mathbf{v}_2]_{\mathcal{S}} \;=\; \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

so

$$\underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 \\ -4 & -7 \end{bmatrix}.$$

Okay, so it's really easy when we are "converting" things into the standard basis.

Now we compute $\underset{\mathcal{B}\leftarrow\mathcal{S}}{P}$. We have

$$\mathbf{e}_1 \;=\; -7\mathbf{v}_1 + 4\mathbf{v}_2, \qquad \mathbf{e}_2 \;=\; -2\mathbf{v}_1 + \mathbf{v}_2 \; .$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{S}}{P} = \; [[\mathbf{e}_1]_{\mathcal{B}} | [\mathbf{e}_2]_{\mathcal{B}}] \;=\; \begin{bmatrix} -7 & -2 \\ 4 & 1 \end{bmatrix}.$$

This illustrates the general principle

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

Example 2: Let $\mathcal{B} = \{(1,-4),\ (2,-7)\}$ as in the previous example and let

$$\mathcal{C} = \left\{\begin{bmatrix} 5 \\ -18 \end{bmatrix}, \begin{bmatrix} 7 \\ -25 \end{bmatrix}\right\}.$$

Since

$$\begin{bmatrix} 5 \\ -18 \end{bmatrix} = \mathbf{v}_1 + 2\mathbf{v}_2, \qquad \begin{bmatrix} 7 \\ -25 \end{bmatrix} = \mathbf{v}_1 + 3\mathbf{v}_2,$$

we have

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

Since

$$\mathbf{v}_1 = 3\cdot(5,-18) - 2\cdot(7,-25), \qquad \mathbf{v}_2 = (-1)\cdot(5,-18) + (7,-25),$$

we have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

which, as we expected, is the inverse of $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

Example 3: If $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{C}$ are three bases for a finite-dimensional vector space V, then we observe a relationship between the various change-of-coordinates matrices between them.

Intuitively, it is clear that changing coordinates from basis $\mathcal{B}$ to basis $\mathcal{S}$ and then changing from there to coordinates relative to basis $\mathcal{C}$ is equivalent to changing coordinates directly from basis $\mathcal{B}$ to basis $\mathcal{C}$:

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{S}}{P}\,[\mathbf{x}]_{\mathcal{S}} = \underset{\mathcal{C}\leftarrow\mathcal{S}}{P}\left(\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}\,[\mathbf{x}]_{\mathcal{B}}\right) = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\mathbf{x}]_{\mathcal{B}}.$$

The point here is that we have the fundamental equation

$$\underset{\mathcal{C}\leftarrow\mathcal{S}}{P}\ \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$$

for any three bases $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{C}$.

Now this gives us an algorithm to find any change-of-basis matrix when there is an easy-to-use natural basis available to us. If $\mathcal{S}$ is some kind of "standard" basis (as we have for $\mathbb{R}^n$, $\mathbb{P}_n$ and $M_{m\times n}$) for V, then computing $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ is trivial. So we need only compute

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{S}}{P}\ \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\right)^{-1}\ \underset{\mathcal{S}\leftarrow\mathcal{B}}{P}\ .$$

We can acheive this by row reducing the partitioned matrix

$$\begin{aligned} \left[\underset{\mathcal{S}\leftarrow\mathcal{C}}{P} \,\middle|\, \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \right] &\sim \left[I \,\middle|\, \left(\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}\right)^{-1} \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \right] \\ &= \left[I \,\middle|\, \underset{\mathcal{C}\leftarrow\mathcal{S}}{P}\,\underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \right] \\ &= \left[I \,\middle|\, \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right] \end{aligned}$$

for any three bases $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{C}$.

In our example, $\mathcal{B} = \{(1,-4),\ (2,-7)\}$, $\mathcal{S} = \{(1,0),\ (0,1)\}$ and $\mathcal{C} = \{(5,-18),\ (7,-25)\}$. we have

$$\underset{\mathcal{S}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 7 \\ -18 & -25 \end{bmatrix}, \qquad \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 \\ -4 & -7 \end{bmatrix}$$

and, indeed,

$$\begin{aligned} \left[\underset{\mathcal{S}\leftarrow\mathcal{C}}{P} \,\middle|\, \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \right] &= \left[\begin{array}{cc|cc} 5 & 7 & 1 & 2 \\ -18 & -25 & -4 & -7 \end{array}\right] \\ &\sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right] \\ &= \left[I \,\middle|\, \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \right] \end{aligned}$$

as expected.

Now let's use these tools to look at other examples.

Example 4: In the vector space $M_{2\times 2}$, consider the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

We want to compute $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

We do this using the standard basis for $M_{2\times 2}$:

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

It is easy to compute

$$\underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\mathcal{S}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now we use row reduction:

$$\left[\begin{array}{c|c} \underset{\mathcal{S}\leftarrow\mathcal{C}}{P} & \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 3 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

We can also invert the matrices $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$ to obtain

$$\underset{\mathcal{B}\leftarrow\mathcal{S}}{P} = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & -3 \\ 0 & 6 & -4 & 0 \\ 12 & -6 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\mathcal{C}\leftarrow\mathcal{S}}{P} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example 5: In the vector space $\mathbb{P}_3$, consider the two bases

$$\mathcal{B} = \left\{1 + 2t + 3t^2 + 4t^3,\ \ 1 + 2t + 3t^2,\ \ 1 + 2t,\ \ 1\right\}$$

and

$$\mathcal{C} = \left\{t^3,\ \ t^2 + t^3,\ \ t + t^2 + t^3,\ \ 1 + t + t^2 + t^3\right\}.$$

Let's compute the change-of-coordinates matrices $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

We do this using the standard basis for $\mathbb{P}_3$:

$$\mathcal{S} = \left\{1,\ t,\ t^2,\ t^3\right\}.$$

These change-of-coordinates matrices are automatic:

$$\underset{\mathcal{S}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\mathcal{S}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Again we use row reduction to find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$:

$$\left[\begin{array}{c|c} \underset{\mathcal{S}\leftarrow\mathcal{C}}{P} & \underset{\mathcal{S}\leftarrow\mathcal{B}}{P} \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 3 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|rrrr} 1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \end{array}\right].$$

The inverses of the matrices $\underset{\mathcal{S}\leftarrow\mathcal{B}}{P}$ and $\underset{\mathcal{S}\leftarrow\mathcal{C}}{P}$ give us

$$\underset{\mathcal{B}\leftarrow\mathcal{S}}{P} = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & -3 \\ 0 & 6 & -4 & 0 \\ 12 & -6 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \underset{\mathcal{C}\leftarrow\mathcal{S}}{P} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

just as in the case of vector spaces of matrices.

The nature of the objects becomes irrelevant. Once we have a standard basis, any four-dimensional vector space looks just like $\mathbb{R}^4$ and we can use ordinary matrix theory to compute in that vector space.