Linear Algebra
C Term, Sections C01-C04
W. J. Martin
March 2, 2010

Sample Solutions
Linear Algebra Assignment 7

Problem 1: Consider the vector space $\mathbb{P}_3$ and the two ordered bases

$$\mathcal{B} = \left\{1,\ t+1,\ t^2+2t+1,\ t^3+3t^2+3t+1\right\}$$

and

$$\mathcal{C} = \left\{t^3,\ t^3-1,\ t^3-t,\ t^3-t^2\right\}$$

for this space.

For each of the following vectors, find the *coordinate vector* of it relative to each of the bases $\mathcal{B}$ and $\mathcal{C}$. Show your work.

(i) $\mathbf{u} = 3 + 3t + t^2$ (find $[\mathbf{u}]_{\mathcal{B}}$ and $[\mathbf{u}]_{\mathcal{C}}$)

(ii) $\mathbf{v} = t^3$ (find $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{C}}$)

(iii) $\mathbf{w} = 1 + t + t^2$ (find $[\mathbf{w}]_{\mathcal{B}}$ and $[\mathbf{w}]_{\mathcal{C}}$)

Solution: **(i)** By definition of the coordinate vector, we have

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{u} = r_1\mathbf{b}_1 + r_2\mathbf{b}_2 + r_3\mathbf{b}_3 + r_4\mathbf{b}_4.$$

The equation

$$3 + 3t + 1t^2 + 0t^3 \; = \; r_1(1) + r_2(1+t) + r_3(1+2t+t^2) + r_4(1+3t+3t^2+t^3)$$

gives us the linear system

$$\begin{array}{rcrrrr} 3 & = & r_1+ & r_2+ & r_3+ & r_4 \\ 3 & = & & r_2+ & 2r_3+ & 3r_4 \\ 1 & = & & & r_3+ & 3r_4 \\ 0 & = & & & & r_4 \end{array}$$

with these implications

$$\Rightarrow\ r_4 = 0\ \Rightarrow\ r_3 = 1\ \Rightarrow\ r_2 = 1\ \Rightarrow\ r_1 = 1.$$

So we have

$$[\mathbf{u}]_{\mathcal{B}} = [1,\ 1,\ 1,\ 0]^\top .$$

Now we do the same thing for basis $\mathcal{C}$.

$$[\mathbf{u}]_{\mathcal{C}} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{u} = s_1\mathbf{c}_1 + s_2\mathbf{c}_2 + s_3\mathbf{c}_3 + s_4\mathbf{c}_4.$$

The equation

$$3 + 3t + 1t^2 + 0t^3 \ = \ s_1(t^3) + s_2(t^3 - 1) + s_3(t^3 - t) + s_4(t^3 - t^2)$$

gives us the linear system

$$\begin{aligned} 3 &= \quad -s_2 \\ 3 &= \quad\quad -s_3 \\ 1 &= \quad\quad\quad -s_4 \\ 0 &= s_1 + \ s_2 + \ s_3 + \ s_4 \end{aligned}$$

which quickly gives us

$$[\mathbf{u}]_{\mathcal{C}} = [7,\ -3,\ -3,\ -1]^\top .$$

(ii) Now we carry out the same process for $\mathbf{v}$ to find

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \qquad [\mathbf{v}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

(iii) . . . and for $\mathbf{w}$ to determine

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad [\mathbf{w}]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

Problem 2: (a) Derive the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are the bases given in Problem 1.

Solution: To do this, we build a matrix with four rows and eight columns. The vectors on the left are the coordinate vectors of the vectors in basis $\mathcal{C}$ while the vectors on the right are the coordinate vectors of the vectors in $\mathcal{B}$. Once we have this matrix, we row reduce it to obtain the desired matrix on the righthand side:

$$\left[\ P_{\mathcal{C}} \mid P_{\mathcal{B}}\ \right] \sim \left[\ I \ \Big|\ \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\ \right].$$

$$\left[\begin{array}{cccc|cccc} 0 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & 4 & 8 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -3 \end{array}\right].$$

So

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 4 & 8 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -3 \end{bmatrix}.$$

(b) Using part (a), derive the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

Solution: We use the fact that

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1}.$$

Since

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 4 & 8 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7 & -8 & -8 & -8 \\ 0 & 1 & 0 & 0 & -6 & -6 & -7 & -7 \\ 0 & 0 & 1 & 0 & -3 & -3 & -3 & -4 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right],$$

we have

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} -7 & -8 & -8 & -8 \\ -6 & -6 & -7 & -7 \\ -3 & -3 & -3 & -4 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Problem 3: For each of the following matrices, compute the characteristic equation of the given matrix and use this to find all eigenvalues of the matrix (including multiplicities).

(a) $A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$

Solution: We compute

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -5 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10$$

so that our characteristic equation is $\lambda^2 - 5\lambda - 6 = 0$, i.e., $(\lambda - 6)(\lambda + 1) = 0$. So the eigenvalues are 6 and -1, each with multiplicity one.

(b) $B = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$

Solution: Again, we compute

$$|B - \lambda I| = \begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda) + 1$$

so that our characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0,$$

i.e., $(\lambda + 2)^2 = 0$. So the only eigenvalue is $\lambda = -2$, with multiplicity two.

(c) $C = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Solution: Since C is upper triangular, the determinant of $C - \lambda I$ is just the product of its entries along the diagonal:

$$|C - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & -1 & 3 \\ 0 & 3-\lambda & 1 & 1 \\ 0 & 0 & 4-\lambda & 6 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(4-\lambda)(4-\lambda)$$

so that our characteristic equation is

$$(2-\lambda)(3-\lambda)(4-\lambda)^2 = 0$$

and the eigenvalues are $\lambda_1 = \lambda_2 = 4$ (multiplicity two) and $\lambda_3 = 3$, $\lambda_4 = 2$ (each with multiplicity one).

(d) $D = \begin{bmatrix} 1 & 0 & -2 \\ 5 & 0 & 0 \\ -3 & 0 & 2 \end{bmatrix}$

Solution: We compute

$$|D - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 5 & -\lambda & 0 \\ -3 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(-\lambda)(2-\lambda) + 0 + 0 + 6\lambda - 0 - 0$$

so that our characteristic equation is $\lambda^3 - 3\lambda^2 - 4\lambda = 0$, or $\lambda(\lambda+1)(\lambda-4) = 0$. We find three eigenvalues all of multiplicity one:

$$\lambda_1 = 4, \qquad \lambda_2 = 0, \qquad \lambda_3 = -1.$$

Problem 4: For each matrix, find a basis for the eigenspace V_λ where $\lambda = 5$.

(a) $A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Solution: The matrix

$$A - 5I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is already in reduced row echelon form. So a basis for its null space is

$$\mathcal{B} = \left\{ [0, 0, 1, 0]^\top, \quad [0, 0, 0, 1]^\top \right\}.$$

This is the desired basis for the eigenspace V_λ.

(b) $B = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Solution: We row reduce

$$A - 5I = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(We see only one free variable, even though the algebraic multiplicity is two.) A basis for the null space is

$$\mathcal{B} = \left\{ [1, -1, 1, 0]^\top \right\}$$

so the eigenspace V_λ has dimension only one.

(c) $C = \begin{bmatrix} 5 & 54 & -8 & -46 \\ 0 & -19 & 4 & 20 \\ 0 & -30 & 5 & 30 \\ 0 & -18 & 4 & 19 \end{bmatrix}$

Solution: We row reduce

$$A - 5I = \begin{bmatrix} 0 & 54 & -8 & -46 \\ 0 & -24 & 4 & 20 \\ 0 & -30 & 0 & 30 \\ 0 & -18 & 4 & 14 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the null space is

$$\mathcal{B} = \left\{ [1,0,0,0]^\top, \quad [0,1,1,1]^\top \right\}$$

so the eigenspace V_λ has dimension two.

Problem 5: Diagonalize the following matrix:

$$A = \begin{bmatrix} 17 & -12 & 36 \\ 24 & -19 & 48 \\ 2 & -2 & 3 \end{bmatrix}$$

(That is, find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.)

Solution: We are given the hint in class that the sum of the first two columns might be interesting. We set $\mathbf{v}_1 = (1,1,0)$ and compute

$$A\mathbf{v}_1 = \begin{bmatrix} 17 & -12 & 36 \\ 24 & -19 & 48 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}.$$

So we see that $\lambda_1 = 5$ is one of our eigenvalues.

Now the trace and determinant should give us enough evidence to recover the other two eigenvalues. We have

$$\operatorname{trace} \mathrm{A} = 17 + (-19) + 3 = 1,$$

so $\lambda_2 + \lambda_3 = -4$. Also, we have

$$\det A = \lambda_1 \lambda_2 \lambda_3 = 15$$

so we work out $\lambda_2 = -1$ and $\lambda_3 = -3$.

Then we solve $(A - 5I)\mathbf{v}_1 = \mathbf{0}$, $(A + I)\mathbf{v}_2 = \mathbf{0}$ and $(A + 3I)\mathbf{v}_3 = \mathbf{0}$ to find eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

that satisfy

$$A\mathbf{v}_1 = 5\mathbf{v}_1, \qquad A\mathbf{v}_2 = -\mathbf{v}_2, \qquad A\mathbf{v}_3 = -3\mathbf{v}_3.$$

So if we take

$$P = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

then we have

$$AP = PD$$

or

$$A = PDP^{-1}$$

which is exactly what we need.