Linear Algebra
C Term, Sections C01-C04
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Sample Solutions

Linear Algebra Assignment 6

Problem 1: We must find bases for the row space, column space, and null space of the following two matrices.

(a) $A = \begin{bmatrix} 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, **(b)** $A = \begin{bmatrix} 1 & 1 & 0 & 3 & 12 & -7 \\ -1 & 2 & -6 & 1 & 0 & 3 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 & 9 & -12 \end{bmatrix}$

Solution: For part (a), the matrix is already in row reduced form. A basis for the row space is obtained by taking the non-zero rows in the r.r.e.f.:

$$\text{Row}\, A = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\}.$$

A basis for the column space is obtained by taking the columns of the *original* matrix A where its r.r.e.f. has pivots. Of course, in this simple case, the two matrices are the same. So

$$\text{Col}\, A = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The null space of A is just the set of solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Since the matrix is already in nice form, we easily read off the solutions:

$$x_1 = r, \quad x_2 = -3s + 2t, \quad x_3 = s, \quad x_4 = -5t, \quad x_5 = t, \quad (r, s, t \in \mathbb{R}).$$

So putting this in parametric vector form, we obtain the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$

and we observe that each of these vectors is orthogonal to each vector in Row$A$.

Problem 1, Part (b): First we need to row reduce the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 & 12 & -7 \\ -1 & 2 & -6 & 1 & 0 & 3 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 & 9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 & 12 & -7 \\ 0 & 3 & -6 & 4 & 12 & -4 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 & 9 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 & 12 & -7 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 3 & 9 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 & 5 & 5 \\ 0 & 1 & -2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now a basis for the row space is obtained by taking the non-zero rows in the r.r.e.f.:

$$\text{Row}\, A = \text{Span}(\mathcal{B}), \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \\ -4 \end{bmatrix} \right\}.$$

A basis for the column space is obtained by taking the columns of the *original* matrix A where its r.r.e.f. has pivots:

$$\text{Col}\, A = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

To find a basis for the null space, we describe all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in parametric vector form. The solutions are

$$\mathbf{x} = r \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -4 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{where } r,\ s \text{ and } t \text{ are any scalars.}$$

By pulling off the vectors corresponding to the various free parameters, we find a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Problem 2: We have the linear transformation T, that takes 2×3 matrices as inputs and gives polynomials of degree three or less as outputs, given by

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = (4a - 2b - 4c - 4d + 14f) + (a + b - c + 2d + e - 2f)t$$
$$+ (2a - b - 2c - 2d + 3e - 5f)t^2 + (-a + b + c + 2d - e)t^3.$$

(a) We want to find a basis for the kernel of T.

Solution: We know that $\operatorname{Ker} T$ is the set of all matrices that map to the zero vector in $\mathbb{P}_3$, which is $\mathbf{0} = 0 + 0t + 0t^2 + 0t^3$. We have

$$A \in \operatorname{Ker} T \Leftrightarrow T(A) = \mathbf{0}$$
$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in \operatorname{Ker} T \Leftrightarrow (4a - 2b - 4c - 4d + 14f) + (a + b - c + 2d + e - 2f)t +$$
$$(2a - b - 2c - 2d + 3e - 5f)t^2 + (-a + b + c + 2d - e)t^3 = \mathbf{0}$$

$$\Leftrightarrow (4a - 2b - 4c - 4d + 14f) + (a + b - c + 2d + e - 2f)t +$$
$$(2a - b - 2c - 2d + 3e - 5f)t^2 + (-a + b + c + 2d - e)t^3 = 0 + 0t + 0t^2 + 0t^3$$
$$\Leftrightarrow \begin{cases} 4a - 2b - 4c - 4d + 14f = 0 \\ a + b - c + 2d + e - 2f = 0 \\ 2a - b - 2c - 2d + 3e - 5f = 0 \\ -a + b + c + 2d - e = 0 \end{cases}$$

So we have a linear system with four equations and six unknowns. We easily reduce this to the equivalent but simpler system

$$\begin{cases} a - c + 3f = 0 \\ b + 2d - f = 0 \\ e - 4f = 0 \end{cases}$$

So we have the following basis for the kernel:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \right\}.$$

(b) Find a basis for the range of T.

Solution: We apply the idea that we use for $\mathbb{R}^n$: the range corresponds to the column space of the matrix

$$\begin{bmatrix} 4 & -2 & -4 & -4 & 0 & 14 \\ 1 & 1 & -1 & 2 & 1 & -2 \\ 2 & -1 & -2 & -2 & 3 & -5 \\ -1 & 1 & 1 & 2 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis for the range of T is given by

$$\left\{4+t+2t^2-t^3,\quad -2+t-t^2+t^3,\quad t+3t^2-t^3\right\}.$$

Problem 3: Using the ordinary dot product, we have to find orthonormal bases for two planes in $\mathbb{R}^4$.

(a) Find an orthonormal basis for the null space of the matrix $Z = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Solution: The null space is the set of all solutions $\mathbf{x}$ to the equation $Z\mathbf{x} = \mathbf{0}$. Since it is already reduced to r.r.e.f., we see that there are two free variables and we can easily read of a basis for its null space: $\mathcal{S} = \{(-1,1,0,0),\ (0,0,-1,1)\}$. Luckily, these vectors are already orthogonal to one another:

$$\langle(-1,1,0,0),\ (0,0,-1,1)\rangle \;=\; (-1)\cdot 0 + 1\cdot 0 + 0\cdot(-1) + 0\cdot 1 = 0.$$

But the dot product of each one with itself is two. So they need to be scaled to unit vectors. Dividing all entries by $\sqrt{2}$, we obtain one possible orthonormal basis for the null space:

$$\mathcal{B} = \left\{ \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right),\ \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}.$$

(There are other valid answers to this question.)

(b) Find an orthonormal basis for the row space of the same matrix Z.

Solution: We know that one basis for this space is given by the rows of its r.r.e.f. Since the matrix is already reduced, we have basis $\mathcal{S} = \{(1,1,0,0),\ (0,0,1,1)\}$. Luckily, these vectors are already orthogonal: their dot product is zero. But the dot product of each one with itself is two. So they need to be scaled to unit vectors. So here is one possible orthonormal basis for the row space:

$$\mathcal{B} = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right),\ \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}.$$

(Of course, there are infinitely many other such bases, but this is the most natural one, given the data.) We immediately observe that not only are these two vectors orthogonal to eachother, but they are also orthogonal to all the vectors in the null space.

(c) Next we consider the Frobenius inner product $\langle A, B\rangle = \mathrm{trace}(\mathrm{AB}^\top)$ on the vector space $M_{2\times 2}$ of two-by-two matrices and we find an orthonormal basis for the subspace of symmetric 2×2 matrices.

Solution: We know that a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is symmetric exactly when $c = b$. So the generic vector in our subspace can be written $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. Writing this in parametric vector form gives us a basis:

$$\mathcal{S} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

We compute inner products

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_1 \rangle &= \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1, \\
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \text{trace} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0, \\
\langle \mathbf{v}_1, \mathbf{v}_3 \rangle &= \text{trace} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, \\
\langle \mathbf{v}_2, \mathbf{v}_2 \rangle &= \text{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2, \\
\langle \mathbf{v}_2, \mathbf{v}_3 \rangle &= \text{trace} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0, \\
\langle \mathbf{v}_3, \mathbf{v}_3 \rangle &= \text{trace} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 1.
\end{aligned}$$

This allows us to quickly find an orthonormal basis by simply scaling the vector $\mathbf{v}_2$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2^{-1/2} \\ 2^{-1/2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(d) A square matrix A is *skew-symmetric* if $A^\top = -A$. Inside the vector space $M_{2\times 2}$ with inner product given above, find an orthonormal basis for the subspace of all skew-symmetric matrices.

Solution: By the same reasoning as above, we find that the generic element of the subspace has form $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$. So we get the basis

$$\mathcal{S} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

and we scale this vector to get orthonormal basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 2^{-1/2} \\ -2^{-1/2} & 0 \end{bmatrix} \right\}.$$

Problem 4: We are asked to project two vectors onto the null space and row space of Z and then two other vectors onto the spaces of symmetric and skew-symmetric matrices.

Solution: We use the results of the previous problem to compute these projections.
(a) Let $\mathcal{N}$ be the null space of the matrix Z studied in Problem 3(a) and let $\mathcal{R}$ be the rows space of Z, these both being subspaces of $\mathbb{R}^4$. Find the standard matrices (each of size 4×4) for the linear transformations $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{R}}$. Explain.

Solution: We project the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ onto each of these spaces using the basis vectors

$$\mathbf{v}_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \qquad \mathbf{v}_2 = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

For each i, $1 \leq i \leq 4$, we have

$$\pi_{\mathcal{N}}(\mathbf{e}_i) = \langle \mathbf{v}_1, \mathbf{e}_i \rangle \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{e}_i \rangle \mathbf{v}_2,$$

that is,

$$\pi_{\mathcal{N}}(\mathbf{e}_1) = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}, \quad \pi_{\mathcal{N}}(\mathbf{e}_2) = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}, \quad \pi_{\mathcal{N}}(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \pi_{\mathcal{N}}(\mathbf{e}_4) = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

So we stack these together to construct the standard matrix of the linear transformation $\pi_{\mathcal{N}}$:

$$P_{\mathcal{N}} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}.$$

The same process, starting from orthonormal basis

$$\mathcal{B} = \left\{ \mathbf{v}_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right), \ \mathbf{v}_4 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$$

and the expression given in the homework assignment

$$\pi_{\mathcal{R}}(\mathbf{e}_i) = \langle \mathbf{v}_3, \mathbf{e}_i \rangle \mathbf{v}_3 + \langle \mathbf{v}_3, \mathbf{e}_i \rangle \mathbf{v}_3,$$

we compute the standard matrix

$$P_{\mathcal{R}} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

(Cool fact: the vector space $V = \mathbb{R}^4$ has an orthogonal decomposition into these two planes: $V = \mathcal{N} \perp \mathcal{R}$. This is reflected in the fact that $P_\mathcal{N}$ and $P_\mathcal{R}$ sum to the 4×4 identity matrix.)

(b) In $\mathbb{R}^4$, consider $\mathbf{u} = (0, \sqrt{2}, 0, \sqrt{2})$. Project $\mathbf{u}$ onto the row space of Z and also onto the null space of Z. Show your work.

Solution: We compute

$$P_\mathcal{R}\mathbf{u} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$P_\mathcal{N}\mathbf{u} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

(c) In the same manner, project the vector $\mathbf{v} = (3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}, 6\sqrt{2})$ onto the row space of Z and also onto the null space of Z. Show your work.

Solution: Since we computed the matrices in part (a), we just have to multiply:

$$P_\mathcal{R}\mathbf{v} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} \\ 4\sqrt{2} \\ 5\sqrt{2} \\ 6\sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 7 \\ 7 \\ 9 \\ 9 \end{bmatrix}$$

and

$$P_\mathcal{N}\mathbf{u} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} \\ 4\sqrt{2} \\ 5\sqrt{2} \\ 6\sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix},$$

the same projection as for $\mathbf{u}$.

(d) We use a bit of cleverness to project the vectors

$$\mathbf{u} = \begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}.$$

onto the subspace of symmetric matrices and also onto the subspace of skew-symmetric matrices.

Solution: We know that the projection of $\mathbf{u}$ onto the space of symmetric matrices is the unique symmetric matrix which is closest to $\mathbf{u}$: that is, if $\pi(\mathbf{u}) = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, then the function

$$f(a,b,d) = (3-a)^2 + (4-b)^2 + (6-b)^2 + (5-d)^2$$

is minimized at $\pi(\mathbf{u})$. We also know that the projections onto symmetric and skew-symmetric matrices must sum to $\mathbf{u}$ itself; so the projection onto the space of skew-symmetric matrices (let's call it $\pi^\dagger(\mathbf{u})$) is equal to $\mathbf{u} - \pi(\mathbf{u})$. Thus we have projections

$$\pi(\mathbf{u}) = \begin{bmatrix} 3 & 5 \\ 5 & 5 \end{bmatrix} \qquad \text{and} \qquad \pi^\dagger(\mathbf{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ ;$$

$$\pi(\mathbf{v}) = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix} \qquad \text{and} \qquad \pi^\dagger(\mathbf{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $\mathbf{v}$ is already a symmetric matrix.

Problem 5: Our last projections are in the vector space

$$V = \{ \ f \ \mid \ f : [0, 2\pi] \to \mathbb{R}, \quad f \text{ piecewise continuous } \}$$

of all piecewise continuous real-valued functions defined on the interval $[0, 2\pi]$. (Note the error in the problem statement: if I don't assume the functions are piecewise continuous — or at least nicely behaved in some way – I have no guarantee that the given inner product is defined!) For this space, we have the inner product

$$\langle f, g \rangle := \int_0^{2\pi} f(x) g(x) \, dx.$$

(a) Let W be the subspace of V spanned by the twelve vectors

$$\cos(x), \sin(x), \quad \cos(2x), \sin(2x), \quad \ldots, \cos(6x), \sin(6x) \ .$$

Find an orthonormal basis for this subspace.

Solution: We know the following integrals from single-variable calculus (or from MAPLE):

$$\int_0^{2\pi} \cos(mx)\cos(nx)\, dx = \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_0^{2\pi} \cos(mx)\sin(nx)\, dx = 0. \qquad \text{for any values of } m \text{ and } n.$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)\, dx = \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{otherwise.} \end{cases}$$

So the vectors

$$\{\cos x, \sin x,\ \cos 2x, \sin 2x,\ \cos 3x, \sin 3x,\ \cos 4x, \sin 4x,\ \cos 5x, \sin 5x,\ \cos 6x, \sin 6x\}$$

are already all orthogonal to eachother. We obtain an orthonormal basis by scaling each one by $1/\sqrt{\pi}$:

$$\mathcal{B} = \{\mathbf{v}_1, v_2, \ldots, \mathbf{v}_{12}\}$$

where

$$\mathbf{v}_k = \begin{cases} \frac{1}{\sqrt{\pi}} \cos(kx) & 1 \le k \le 6; \\ \frac{1}{\sqrt{\pi}} \sin((k-6)x) & 7 \le k \le 12. \end{cases}$$

(b) Consider the function

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x < \pi; \\ \ \ 1 & \text{if } \pi \le x \le 2\pi. \end{cases}$$

Find the projection of $f(x)$ onto subspace W.

Solution: We know from calculus that $\int_0^\pi \cos(kx)\,dx = 0$ and $\int_\pi^{2\pi} \cos(kx)\,dx = 0$ for $k > 0$, so

$$\langle \mathbf{v}_k, f\rangle = 0 \qquad \text{for } 1 \le k \le 6.$$

We also know (or can use MAPLE to recall) that

$$\int_0^\pi \sin(kx)\,dx = \int_\pi^{2\pi} \sin(kx)\,dx = \begin{cases} -2/k & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

So we find

$$\langle \mathbf{v}_k, f\rangle = 0 \qquad \text{for } k = 8, 10, 12$$

and

$$\langle \mathbf{v}_7, f\rangle = -\frac{4}{\sqrt{\pi}}, \qquad \langle \mathbf{v}_9, f\rangle = -\frac{4}{3\sqrt{\pi}}, \qquad \langle \mathbf{v}_1 1, f\rangle = -\frac{4}{5\sqrt{\pi}}.$$

So we have our orthogonal projection

$$\pi_W(f) = \sum_{k=1}^{12} \langle \mathbf{v}_k, f\rangle \mathbf{v}_k \ = \ -\frac{4}{\pi}\sin x - \frac{4}{3\pi}\sin(3x) - \frac{4}{5\pi}\sin(5x).$$

Figure 1 gives a plot of this best approximation to the step function f.

(c) Consider the function

$$g(x) = \begin{cases} 2x - \pi & \text{if } 0 \le x < \pi; \\ 2x - 3\pi & \text{if } \pi \le x \le 2\pi. \end{cases}$$

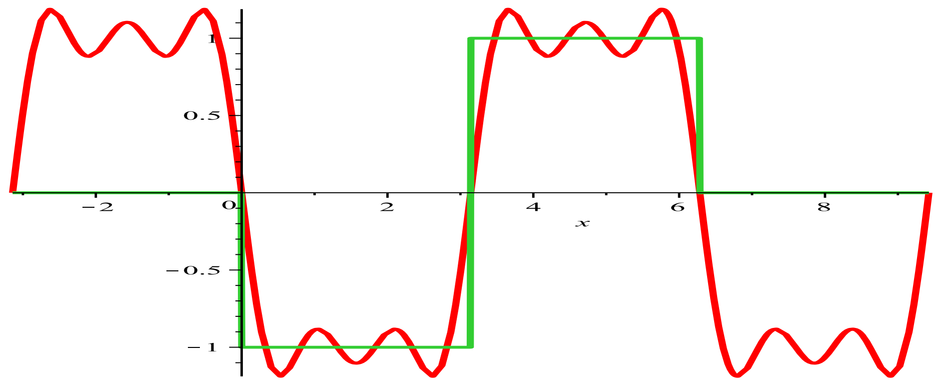

Figure 1: The sine functions repeat the square wave beyond its domain.

Find the projection of $g(x)$ onto subspace W.

Solution: Ugh! In addition to the above integration rules, we need to remember how we integrated $x\cos(kx)$ and $x\sin(kx)$. Or we can just look in our calculus book, or just ask MAPLE. We find

$$\int_0^{2\pi} x\cos(kx)\,dx = 0 \qquad (k = 1, 2, 3, \ldots);$$
$$\int_0^{2\pi} x\sin(kx)\,dx = -\frac{2\pi}{k} \qquad (k = 1, 2, 3, \ldots).$$

We can use the previous computation! Since all dot products are linear transformations of either argument, we exploit the connection

$$g(x) = 2x + \pi - f(x)$$

to get

$$\langle \mathbf{v}_k, g\rangle = 2\langle \mathbf{v}_k, x\rangle + \pi\langle \mathbf{v}_k, 1\rangle - \langle \mathbf{v}_k, f\rangle\ .$$

Since $\langle \mathbf{v}_k, 1\rangle = 0$ for all k (from above), we get

$$\langle \mathbf{v}_k, g\rangle = 0 \qquad \text{for } k \neq 8, 10, 12$$

and

$$\langle \frac{1}{\sqrt{\pi}}\sin(kx), g(x)\rangle = -\frac{4\sqrt{\pi}}{k}$$

for k even. So

$$\pi_W(g) = -2\sin(2x) - \sin(4x) - \frac{2}{3}\sin(6x).$$

Figure 2 gives a plot of $g(x)$ and its projection onto W.

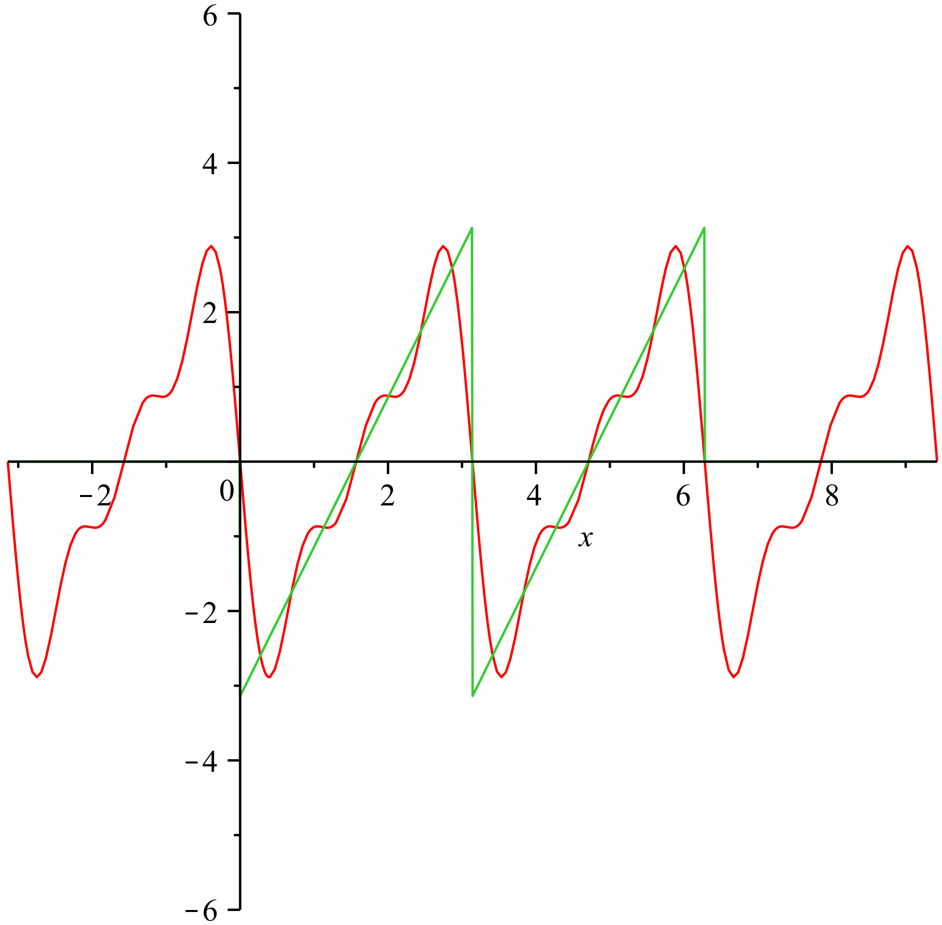

Figure 2: The sine functions are trying to model a sawtooth.