Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Assignment 6

DUE DATE: Wednesday, February 24, 4:30pm. Place in mail bin labelled "MA2071" in Room SH108. There, you will find a separate folder for each section. (Be sure to write your name and your section number on the 1st page.)

Please recall Professor Martin's rules for Linear Algebra assignments.

Please complete the following five problems.

1. In each part, find
   - a basis for the row space of A
   - a basis for the column space of A
   - a basis for the null space of A

(a) $A = \begin{bmatrix} 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, **(b)** $A = \begin{bmatrix} 1 & 1 & 0 & 3 & 12 & -7 \\ -1 & 2 & -6 & 1 & 0 & 3 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 0 & 0 & 0 & 3 & 9 & -12 \end{bmatrix}$

2. Consider the linear transformation $T : M_{2\times 3} \to \mathbb{P}_3$ given by

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = (4a - 2b - 4c - 4d + 14f) + (a + b - c + 2d + e - 2f)t + (2a - b - 2c - 2d + 3e - 5f)t^2 + (-a + b + c + 2d - e)t^3.$$

(a) Find a basis for the kernel of T.

(b) Find a basis for the range of T.

[HINT: Rewrite this as a linear transformation $T' : \mathbb{R}^6 \to \mathbb{R}^4$ of the form $T'(\mathbf{x}) = A\mathbf{x}$ for some 4×6 matrix A. But remember to write your answers in the form of the vectors spaces in the question.]

3. For this course, we will write the dot product of two vectors in $\mathbb{R}^n$ as a special case of a general "inner product": for $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, instead of $\mathbf{u} \cdot \mathbf{v}$, we'll write

$$\langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \ .$$

Definition: An *orthonormal basis* for an inner product space (i.e., a vector space with a "positive definite" inner product like this one) is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in which all vectors are length one and they are pairwise "perpendicular":

$$\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1 \ \ (1 \leq i \leq k), \qquad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \ \ (\text{whenever } i \neq j).$$

(a) Find an orthonormal basis in $\mathbb{R}^4$ for the null space of the matrix $Z = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(b) Find an orthonormal basis for the row space of the same matrix Z. Do you notice any connection to part (a)?

(c) Now consider the vector space $M_{2\times 2}$ of all 2×2 matrices with real entries. Our inner product on this space is

$$\langle A, B \rangle = \text{trace}(\text{AB}^\top)$$

where the *trace* of a square matrix is the sum of its diagonal entries.

Find an orthonormal basis for the subspace of all symmetric 2×2 matrices. (Recall that a square matrix A is *symmetric* if $A^\top = A$.)

(d) A square matrix A is *skew-symmetric* if $A^\top = -A$. Inside the vector space $M_{2\times 2}$ with inner product given above, find an orthonormal basis for the subspace of all skew-symmetric matrices.

The last two problems deal with orthogonal projections. Let's first discuss these for $V = \mathbb{R}^n$. If W is a subspace of V, then it is possible to extend any basis for W to a basis for V. Moreover, we can do this in such a way that all of the new vectors in the basis are orthogonal to W.

Suppose we can find an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ for W. In class, we explained how, for any $\mathbf{v}$ in V, the closest vector in W to $\mathbf{v}$ is its projection

$$\pi_W(\mathbf{v}) = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + \cdots c_k \mathbf{w}_k$$

defined by $c_i = \langle \mathbf{w}_i, \mathbf{v} \rangle$ for $1 \leq i \leq k$.

This same idea applies to any finite-dimensional subspace of any inner product space.

4. Using the results of the previous problem, compute the following projections.

(a) Let $\mathcal{N}$ be the null space of the matrix Z studied in Problem 3(a) and let $\mathcal{R}$ be the rows space of Z, these both being subspaces of $\mathbb{R}^4$. Find the standard matrices (each of size 4×4) for the linear transformations $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{R}}$. Explain.

(b) In $\mathbb{R}^4$, consider $\mathbf{u} = (0, \sqrt{2}, 0, \sqrt{2})$. Project $\mathbf{u}$ onto the row space of Z and also onto the null space of Z. Show your work.

(c) In the same manner, project the vector $\mathbf{v} = (3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}, 6\sqrt{2})$ onto the row space of Z and also onto the null space of Z. Show your work.

(d) In $M_{2\times 2}$, consider

$$\mathbf{u} = \begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}.$$

Project the two vectors $\mathbf{u}$ and $\mathbf{v}$ onto the subspace of symmetric matrices and also onto the subspace of skew-symmetric matrices, where these subspaces were studied in Problem 3(c,d). Show your work.

5. Finally, we study the vector space $V = \{f \mid f : [0, 2\pi] \to \mathbb{R}\}$ of all real-valued functions defined on the interval $[0, 2\pi]$. The addition in this vector space is usual addition of functions and scalar multiplication is as usual also: $f + g$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for $x \in [0, 2\pi]$ and cf is the function $(cf)(x) = c(f(x))$ for $x \in [0, 2\pi]$.

The inner product we consider on this space is similar to the one we considered in Assignment 1:

$$\langle f(x), g(x) \rangle := \int_0^{2\pi} f(x)g(x)\,dx.$$

(a) Let W be the subspace of V spanned by the twelve vectors

$$\cos(x), \sin(x), \quad \cos(2x), \sin(2x), \quad \ldots, \cos(6x), \sin(6x)\ .$$

Find an orthonormal basis for this subspace. *[HINT: You don't have to do too much to adjust the above basis. Use standard formulas for integrals.]*

(b) Consider the function

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x < \pi; \\ 1 & \text{if } \pi \le x \le 2\pi. \end{cases}$$

Find the projection of $f(x)$ onto subspace W. Plot it using any appropriate software.

(c) Consider the function

$$g(x) = \begin{cases} 2x - \pi & \text{if } 0 \leq x < \pi; \\ 2x - 3\pi & \text{if } \pi \leq x \leq 2\pi. \end{cases}$$

Find the projection of $g(x)$ onto subspace W. Plot it using any appropriate software.