Linear Algebra
C Term, Sections C01-C04
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Sample Solutions

Linear Algebra Assignment 5

Problem 1: Find the 3×3 matrix that produces the following composite 2D transformation by acting on homogeneous coordinates: *Translate by* $(2, -5)$ *and then reflect about the line* $x = -3$.

Solution: Our translation is

$$T = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Our reflection is obtained by conjugating B by A — $R = ABA^{-1}$ — where

$$A = \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right], \qquad B = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

(Note that the top-left block of B is the 2×2 matrix which reflects the plane across the y-axis.) So we have

$$\begin{aligned} R &= ABA^{-1} \\ &= \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -1 & 0 & -6 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \\ RT &= \left[\begin{array}{cc|c} -1 & 0 & -6 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \\ \hline 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -1 & 0 & -8 \\ 0 & 1 & -5 \\ \hline 0 & 0 & 1 \end{array}\right] \end{aligned}$$

Problem 2: Find the 4×4 matrix that produces the following 3D transformation using homogeneous coordinates: *Rotate by* $30°$ *about the line joining* $(0, 1, 0)$ *to* $(2, 1, 0)$*, in a* *clockwise direction when viewed from an observer located at* $(2, 1, 0)$ *looking at the plane* $x = 0$.

Solution: Since the axis of rotation does not pass through the origin, this rotation is not a linear transformation on 3-space. So we'll have to conjugate a well-chosen matrix by a translation. The point $(0, 1, 0)$ is on the axis of rotation, so a translation matrix that moves the origin to this point is

$$A = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

The rotation we now need is one by $30°$ clockwise around the x-axis, looking from the direction of the positive x-axis:

$$B = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

(The x-axis is fixed and $B\mathbf{e}_2$, $B\mathbf{e}_3$ stay in the yz-plane.)

$$\begin{aligned} R &= ABA^{-1} \\ &= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 0 \\ 0 & -1/2 & \sqrt{3}/2 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \\ R &= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 & 1-\sqrt{3}/2 \\ 0 & -1/2 & \sqrt{3}/2 & 1/2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{aligned}$$

Problem 3: # 12 on p166. We are to multiply three given matrices and verify that this achieves a rotation (in homogeneous coordinates) by angle φ about the origin.

Solution: Let's just do it:

$$\left[\begin{array}{cc|c} 1 & -\tan\frac{\varphi}{2} & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & -\tan\frac{\varphi}{2} & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] =$$

$$\left[\begin{array}{cc|c} 1-\tan\frac{\varphi}{2}\sin\varphi & -\tan\frac{\varphi}{2} & 0 \\ \sin\varphi & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & -\tan\frac{\varphi}{2} & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] =$$

$$\left[\begin{array}{cc|c} (1-\tan\frac{\varphi}{2}\sin\varphi) & (\tan^2\frac{\varphi}{2}\sin\varphi - 2\tan\frac{\varphi}{2}) & 0 \\ (\sin\varphi) & (1-\tan\frac{\varphi}{2}\sin\varphi) & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Now we apply an old half-angle formula for tangent

$$\tan\frac{\varphi}{2} = \frac{\sin\varphi}{1+\cos\varphi} = \frac{1-\cos\varphi}{\sin\varphi}.$$

(I could have just copied it from Wikipedia; luckily, their version is correct!) In the (1,1)-entry, the second expression for $\tan\frac{\varphi}{2}$ allows us to write $\cos\varphi$ there. Same for the (2,2)-entry of the matrix. The (2,1)-entry simplifies using elementary trigonometry:

$$\tan^2\frac{\varphi}{2}\sin\varphi - 2\tan\frac{\varphi}{2} = \frac{(1-\cos\varphi)^2}{\sin\varphi} - \frac{2(1-\cos\varphi)}{\sin\varphi} = \frac{1-2\cos\varphi+\cos^2\varphi-2+2\cos\varphi}{\sin\varphi} =$$

$$\frac{\cos^2\varphi - 1}{\sin\varphi} = \frac{-\sin^2\varphi}{\sin\varphi} = -\sin\varphi.$$

So our product of three matrices simplifies to

$$\left[\begin{array}{cc|c} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ \hline 0 & 0 & 1 \end{array}\right],$$

which is exactly the rotation matrix we wanted. The whole point of this factorization is that multiplying a vector by this last matrix requires four multiplications of real numbers whereas the multiplication of this same vector by the three matrices we started with only requires three multiplications. This 25% savings in computation really adds up when millions of these rotations are performed sequentially in a computer graphics animation.

Problem 4: #18 on p166. We are to first rotate by 30° clockwise about the z-axis (viewed from above the xy-plane and then translate by $(5, -2, 1)$.

Solution: Our first matrix is

$$R = \left[\begin{array}{ccc|c} \sqrt{3}/2 & 1/2 & 0 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Our second matrix is

$$T = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Their product is

$$TR = \left[\begin{array}{ccc|c} \sqrt{3}/2 & 1/2 & 0 & 5 \\ -1/2 & \sqrt{3}/2 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right].$$

Problem 5: For each of the following structures $(V, \oplus, \odot)$, we show that it is **not** a vector space by identifying a specific axiom that is violated.

(a) $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$ together with the operations

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \oplus \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ 0 \end{bmatrix} \qquad \text{and} \qquad r \odot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} ra \\ rb \\ rc \end{bmatrix}$$

SOLUTION: This monster doesn't have a zero vector; so Axiom 4 is violated. Let $\mathbf{u} = (1, 2, 3)$. Suppose $\mathbf{0} = (a, b, c)$. Then, by definition of $\oplus$,

$$\mathbf{u} \oplus \mathbf{0} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1+a \\ 2+b \\ 0 \end{bmatrix}$$

and this cannot equal $\mathbf{u}$ no matter what values we choose for a, b and c since the third coordinates do not match.

(b) $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b > 0 \right\}$ together with the operations

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ bd \end{bmatrix} \qquad \text{and} \qquad r \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ rb \end{bmatrix}.$$

SOLUTION: This beast has a perfectly reasonable vector addition, albeit a bit strange. It's scalar multiplication looks natural but violates the closure law (Axiom 6): let $\mathbf{u} = (2, 3)$ and $r = -5$. Then $r \odot \mathbf{u} = (-10, -15)$ is not a "vector" according to this definition of the set V.

(c) $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a, b \neq 0 \right\}$ together with the operations

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ bd \end{bmatrix} \qquad \text{and} \qquad r \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ rb \end{bmatrix}$$

for $r \neq 0$ and $0 \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution: This animal is complicated. But it has issues too. Since vector "addition" is defined so strangely, it seems natural to look at the distributive law. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad r = 5.$$

Then, according to these rules,

$$\mathbf{u} \oplus \mathbf{v} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \qquad r \odot \mathbf{u} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \qquad r \odot \mathbf{v} = \begin{bmatrix} 15 \\ 20 \end{bmatrix}.$$

So we get $r \odot (\mathbf{u} \oplus \mathbf{v}) = (15, 40)$ on the one hand, and $(r \odot \mathbf{u}) \oplus (r \odot \mathbf{v}) = (75, 200)$ on the other. So Axiom 7 is brutally violated.

(d)

$$V = \left\{ T : \mathbb{R}^2 \to \mathbb{R}^3 \;\mid\; T \text{ is a linear trans.} \right\}$$

together with the operations

$$(T \oplus T')(\mathbf{x}) = T(\mathbf{x}) + T'(\mathbf{x})$$

for $\mathbf{x}$ in $\mathbb{R}^2$ and

$$(r \odot T)\mathbf{x} = T\left(\begin{bmatrix} rx_2 \\ rx_1 \end{bmatrix} \right).$$

Solution: This critter is so close to a vector space. Except for the coordinate flip at the end. That ruins it. Consider the vector T defined by

$$T(\mathbf{x}) = (x_1, x_1, x_2)$$

and look at the vector $1 \odot T$. For $\mathbf{x} \in \mathbb{R}^2$, we have

$$(1 \odot T)(\mathbf{x}) = T(x_2, x_1) = (x_2, x_2, x_1)$$

which is not the same transformation as T: consider $\mathbf{x} = \mathbf{e}_1 = (1, 0)$, then $T(\mathbf{x}) = (1, 1, 0)$ while

$$(1 \odot T)(\mathbf{x}) = (0, 0, 1).$$

So T and $1 \odot T$ are not the same vector: Axiom 10 is violated here.