Linear Algebra
C Term, Sections C01-C04
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Sample Solutions

Linear Algebra Assignment 4

Problem 1: Show that $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ is invertible if and only if $a \neq 0$ and $c \neq 0$ and $f \neq 0$ and, in this case, find A^{-1}.

Solution: We row reduce the augmented matrix, keeping track of invertibility conditions as we go along. All we need to remember is IMT(c); if in the row reduction process, we ever see a row of zeros, then we know that we will have less than three pivots and the matrix will be singular. For example, if $a = 0$, then matrix A has a row of zeros and cannot be invertible. So assume $a \neq 0$.

$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ b & c & 0 & 0 & 1 & 0 \\ d & e & f & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & c & 0 & -\frac{b}{a} & 1 & 0 \\ 0 & e & f & -\frac{d}{a} & 0 & 1 \end{array}\right] \begin{array}{l} \frac{1}{a}(R1) \\ (R2) - \frac{b}{a}(R1) \\ (R3) - \frac{d}{a}(R1) \end{array}$$

Now if $c = 0$, then we will have a row of zeros on the left and less than three pivots. So also assume $c \neq 0$ and proceed

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{b}{ac} & \frac{1}{c} & 0 \\ 0 & 0 & f & -\frac{be-dc}{ac} & -\frac{e}{c} & 1 \end{array}\right] \begin{array}{l} (R1) \\ \frac{1}{c}(R2) \\ (R3) - \frac{e}{c}(R2) \end{array} \qquad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{b}{ac} & \frac{1}{c} & 0 \\ 0 & 0 & 1 & -\frac{be-dc}{acf} & -\frac{e}{cf} & \frac{1}{f} \end{array}\right] \begin{array}{r} (R1) \\ (R2) \\ \frac{1}{f}(R3) \end{array}$$

where we have also been forced to assume that $f \neq 0$ for the same reasons as above.

Thus, we have derived the general inverse of a lower-triangular 3×3 matrix: if $a = 0$ or $c = 0$ or $f = 0$, then no inverse exists. Otherwise, we have

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \qquad A^{-1} = \frac{1}{acf} \begin{bmatrix} cf & 0 & 0 \\ -bf & af & 0 \\ be - dc & -ae & ac \end{bmatrix} \qquad \text{provided } a,\ c,\ f \text{ all nonzero.}$$

Problem 2: We are inverting Pascal's Triangle. We have

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

SOLUTION: In part (a), we compute A^{-1} and B^{-1} as submatrices of C^{-1}. So we only have to row reduce

$$
\begin{aligned}
[C|I] &= \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 1&1&0&0&0&1&0&0\\ 1&2&1&0&0&0&1&0\\ 1&3&3&1&0&0&0&1 \end{array}\right] \\
&\sim \left[\begin{array}{cc|cc|cc|cc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-1&1&0&0\\ \hline 0&2&1&0&-1&0&1&0\\ 0&3&3&1&-1&0&0&1 \end{array}\right] \quad \text{(found } A^{-1} \text{ in top-left corner)} \\
&\sim \left[\begin{array}{ccc|c|ccc|c} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-1&1&0&0\\ 0&0&1&0&1&-2&1&0\\ \hline 0&0&3&1&2&-3&0&1 \end{array}\right] \quad \text{(found } B^{-1} \text{ in top-left corner)} \\
&\sim \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-1&1&0&0\\ 0&0&1&0&1&-2&1&0\\ 0&0&0&1&-1&3&-3&1 \end{array}\right]
\end{aligned}
$$

So, keeping track of the intermediate results, we have

$$
A^{-1} = \begin{bmatrix} 1&0\\ -1&1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1&0&0\\ -1&1&0\\ 1&-2&1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1&0&0&0\\ -1&1&0&0\\ 1&-2&1&0\\ -1&3&-3&1 \end{bmatrix}.
$$

(b) Seeing this pattern, we conjecture that the pattern continues for larger and larger chunks of Pascal's Triangle. And Theorem 2.6(c) tells us that the inverse of $M^\top$ is the transpose of M^{-1}. So we claim

$$
M = \begin{bmatrix} 1&1&1&1&1&1\\ 0&1&2&3&4&5\\ 0&0&1&3&6&10\\ 0&0&0&1&4&10\\ 0&0&0&0&1&5\\ 0&0&0&0&0&1 \end{bmatrix}, \qquad M^{-1} = \begin{bmatrix} 1&-1&1&-1&1&-1\\ 0&1&-2&3&-4&5\\ 0&0&1&-3&6&-10\\ 0&0&0&1&-4&10\\ 0&0&0&0&1&-5\\ 0&0&0&0&0&1 \end{bmatrix}.
$$

This can be easily verified in MAPLE. It is a special case of the binomial identity

$$
\sum_{k=0}^{\infty} (-1)^{k+j} \binom{i}{k} \binom{k}{j} = \delta_{i,j}
$$

for any non-negative integers i and j where the *Kronecker delta*, $\delta_{i,j}$, is the (i,j)-entry of the identity matrix: one if $i = j$ and zero otherwise.

Problem 3: Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 8 \\ 0 & -1 & 9 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ 7 & -8 & 40 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Compute the following eight matrix products:

$$A^2, \quad BD, \quad A\mathbf{u}, \quad B\mathbf{v}, \quad B^\top A, \quad CD, \quad B^\top\mathbf{u}, \quad C\mathbf{v}.$$

SOLUTION: We grind out

$$A^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}, \quad BD = \begin{bmatrix} -1 & -8 \\ 0 & -9 \end{bmatrix}, \quad A\mathbf{u} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \quad B\mathbf{v} = \begin{bmatrix} -13 \\ -8 \end{bmatrix},$$

$$B^\top A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 8 & 43 \end{bmatrix}, \quad CD = \begin{bmatrix} 0 & 0 \\ -2 & 0 \\ -7 & -40 \end{bmatrix}, \quad B^\top\mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ 51 \end{bmatrix}, \quad C\mathbf{v} = \begin{bmatrix} 0 \\ -2 \\ -46 \end{bmatrix}.$$

(b) Use the results of part (a) to compute the product of the partitioned matrices M and N given by

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}, \qquad N = \begin{bmatrix} A & \mathbf{u} \\ D & \mathbf{v} \end{bmatrix}.$$

SOLUTION: We compute

$$MN = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \begin{bmatrix} A & \mathbf{u} \\ D & \mathbf{v} \end{bmatrix} = \begin{bmatrix} (A^2 + BD) & (A\mathbf{u} + B\mathbf{v}) \\ (B^\top A + CD) & (B^\top\mathbf{u} + C\mathbf{v}) \end{bmatrix}.$$

So

$$MN = \left[\begin{array}{cc|c} 0 & 0 & -4 \\ 0 & 0 & 1 \\ \hline 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & 5 \end{array}\right].$$

Problem 4: Show that the partitioned matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, with blocks A_{11} and A_{22} both square, is invertible if and only if both A_{11} and A_{22} are invertible.

SOLUTION: First, suppose that A_{11} and A_{22} are both invertible. Consider the matrix

$$B = \left[\begin{array}{c|c} A_{11}^{-1} & A_{11}^{-1}A_{12}A_{22}^{-1} \\ \hline 0 & A_{22}^{-1} \end{array}\right].$$

We claim that B is the inverse of A. Let's check:

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \left[\begin{array}{c|c} A_{11}^{-1} & A_{11}^{-1}A_{12}A_{22}^{-1} \\ \hline 0 & A_{22}^{-1} \end{array}\right] \\ &= \left[\begin{array}{c|c} A_{11}A_{11}^{-1} & A_{11}\left(A_{11}^{-1}A_{12}A_{22}^{-1}\right) + A_{12}\left(A_{22}^{-1}\right) \\ \hline 0+0 & A_{22}A_{22}^{-1} \end{array}\right] \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

Now the logical converse is: if A is invertible, then both of the square blocks A_{11} and A_{22} must be invertible. To show this, assume that A is invertible and write its inverse – let's call it B – in block form, conformable to the block structure of A:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

We compute

$$\begin{aligned} AB &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21}) & (A_{11}B_{12} + A_{12}B_{22}) \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

since we are assuming that $B = A^{-1}$. This gives us four equations for the blocks of B:

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I \\ A_{11}B_{12} + A_{12}B_{22} &= 0 \\ A_{22}B_{21} &= 0 \\ A_{22}B_{22} &= I \end{aligned}$$

From the last one, we see that B_{22} is the inverse of A_{22}, so A_{22} must be invertible. From the third equation, we can now eliminate the invertible A_{22} as follows

$$\begin{aligned} A_{22}B_{21} &= 0 \\ B_{22}(A_{22}B_{21}) &= B_{22}0 \\ (B_{22}A_{22})B_{21} &= 0 \\ IB_{21} &= 0 \end{aligned}$$

to find that $B_{21} = 0$ must hold. But then the first equation simplifies to $A_{11}B_{11} = I$ which implies that the block A_{11} is also invertible. That's what we wanted. So we're done!

Problem 5(a): If A is an invertible $n \times n$ matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero n-vector $\mathbf{x}$ and some scalar λ, then

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

SOLUTION: We multiply both sides of the above equation on the left by A^{-1}:

$$\begin{aligned} A\mathbf{x} &= \lambda\mathbf{x} \\ A^{-1}(A\mathbf{x}) &= A^{-1}(\lambda\mathbf{x}) \\ (A^{-1}A)\mathbf{x} &= \lambda(A^{-1}\mathbf{x}) \\ I\mathbf{x} &= \lambda(A^{-1}\mathbf{x}) \\ \mathbf{x} &= \lambda(A^{-1}\mathbf{x}) \end{aligned}$$

Now if $\lambda = 0$, then the vector on the right is the zero vector. But we are given that $\mathbf{x} \neq \mathbf{0}$. So we can conclude that $\lambda \neq 0$ and we can divide both sides by λ to get

$$\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$$

which is what we wanted.

(b) Consider the matrix

$$B = \begin{bmatrix} 7/6 & -1/6 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 7 & -1 & 2 \\ 2 & 4 & 4 \\ -3 & 3 & 0 \end{bmatrix}.$$

Find three linearly independent eigenvectors for B and their corresponding eigenvalues.

SOLUTION: We first observe that B is the inverse of the matrix A appearing in Problem 5(b) on Assignment 2:

$$BA = \frac{1}{6} = \frac{1}{6}\begin{bmatrix} 7 & -1 & 2 \\ 2 & 4 & 4 \\ -3 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 4 \\ -3 & 3 & -5 \end{bmatrix} =$$

$$\frac{1}{6}\begin{bmatrix} (7\cdot 2 - 1\cdot 2 + 2\cdot(-3)) & (7\cdot(-1) - 1\cdot(-1) + 2\cdot 3) & (7\cdot 2 - 1\cdot 4 + 2\cdot(-5)) \\ (2\cdot 2 + 4\cdot 2 + 4\cdot(-3)) & (2\cdot(-1) + 4\cdot(-1) + 4\cdot 3) & (2\cdot 2 + 4\cdot 4 + 4\cdot(-5)) \\ (-3\cdot 2 + 3\cdot 2) & (-3\cdot(-1) + 3\cdot(-1)) & (-3\cdot 2 + 3\cdot 4) \end{bmatrix} = I_3.$$

So we know from part (a) that each eigenvector of A is also an eigenvector of B. The solution to Problem 5(b) on Assignment 2 then gives us two of the three eigenvectors that we need:

$$\mathbf{u} = (1, 1, 0), \quad B\mathbf{u} = 1\mathbf{u}, \qquad \mathbf{v} = (2, 0, 1), \quad B\mathbf{v} = 1\mathbf{v}.$$

These are obtained by setting $r = 1, s = 0$ and $r = 0, s = 1$, respectively in the parametric description of the solutions in 5(b). Clearly these two vectors are linearly independent.

Now we have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1$ for these two vectors (these being the reciprocals of the corresponding eigenvalues of A). The **trace** of B is

$$b_{11} + b_{22} + b_{33} = \frac{7}{6} + \frac{4}{6} + \frac{0}{6} = \frac{11}{6}.$$

We are given the general principle that the trace is equal to the sum of the eigenvalues:

$$\frac{11}{6} = \lambda_1 + \lambda_2 + \lambda_3 = \frac{6}{6} + \frac{6}{6} + \lambda_3$$

which gives us the last eigenvalue we need: $\lambda_3 = -1/6$.

We find our last eigenvector by solving the homogeneous linear system $(B - \lambda_3 I)\mathbf{w} = \mathbf{0}$:

$$\begin{aligned}
[B + \frac{1}{6}I|\mathbf{0}] \quad &\sim \quad \left[\begin{array}{rrr|r} 8 & -1 & 2 & 0 \\ 2 & 5 & 4 & 0 \\ -3 & 3 & 1 & 0 \end{array}\right] \\
&\sim \quad \left[\begin{array}{ccc|c} 8 & -1 & 2 & 0 \\ 0 & \frac{21}{4} & \frac{7}{2} & 0 \\ 0 & \frac{21}{8} & \frac{7}{4} & 0 \end{array}\right] \\
&\sim \quad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

So we get (with $x_3 = 3$ to clear the denominators), $\mathbf{w} = (-1, -2, 3)$ and

$$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

is the linearly independent set of three eigenvectors we were looking for.