Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Assignment 4

DUE DATE: Tuesday, February 9, 4:30pm. Place in mail bin labelled "MA2071" in Room SH108. There, you will find a separate folder for each section. PLEASE WRITE YOUR NAME AND SECTION NUMBER ON YOUR ASSIGNMENT.

Please recall Professor Martin's rules for Linear Algebra assignments.

Please complete the following five problems.

1. Let $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ where a, b, c, d, e, f are real numbers. (This is called a *lower triangular matrix*.)

   (a) Show that A is non-singular if and only if $a \neq 0$ and $c \neq 0$ and $f \neq 0$.

   (b) Assuming a, c, f are all non-zero, use row reduction to work out the general form for A^{-1}.

2. **(a)** Showing the steps in your row reduction, find the inverses of the following three matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

   (b) Using this pattern and Theorem 2.6(c), can you guess the form of the inverse of the following matrix?

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 6 & 10 \\ 0 & 0 & 0 & 1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 8 \\ 0 & -1 & 9 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -2 & 0 \\ 7 & -8 & 40 \end{bmatrix},$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Compute the following eight matrix products:

$$A^2, \quad BD, \quad A\mathbf{u}, \quad B\mathbf{v}, \quad B^\top A, \quad CD, \quad B^\top \mathbf{u}, \quad C\mathbf{v}.$$

(b) Use the results of part (a) to compute the product of the partitioned matrices M and N given by

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}, \qquad N = \begin{bmatrix} A & \mathbf{u} \\ D & \mathbf{v} \end{bmatrix}.$$

4. Problem #14 on p139.

5. **(a)** Prove that, if A is an invertible $n \times n$ matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some non-zero n-vector $\mathbf{x}$ and some scalar λ, then

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}.$$

[The solution is only a few lines of derivation, but be very careful in your logic.]

(b) Consider the matrix

$$B = \begin{bmatrix} 7/6 & -1/6 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ -1/2 & 1/2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 7 & -1 & 2 \\ 2 & 4 & 4 \\ -3 & 3 & 0 \end{bmatrix}.$$

Use the following tools to find three linearly independent eigenvectors for B and their corresponding eigenvalues:

- what you learned from part (a) of this problem;
- the solution to Problem 5(b) on Assignment 2;
- FACT: the 3×3 matrix B has exactly three eigenvalues – two of these happen to be equal – and the sum of these eigenvalues is equal to the **trace** of B, which is defined as the sum of the entries down the main diagonal:

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}\,(B) = b_{11} + b_{22} + b_{33}.$$

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