Linear Algebra
C Term, Sections C01-C04
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Sample Solutions
Linear Algebra Assignment 3

1.) For each of the following functions f from $\mathbb{R}^n$ to $\mathbb{R}^m$, show that f is **not** a linear transformation. Be specific! Give explicit vectors and constants (if necessary) which show a part of the definition which fails for this particular function f.

(a) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = -5$.

Solution: Take $\mathbf{u} = 1$ and $\mathbf{v} = 2$. Then $\mathbf{u} + \mathbf{v} = 3$. We compute $f(\mathbf{u}) = -5$, $f(\mathbf{v}) = -5$ and $f(\mathbf{u} + \mathbf{v}) = -5$. Since

$$f(\mathbf{u} + \mathbf{v}) = -5 \neq -10 = f(\mathbf{u}) + f(\mathbf{v}),$$

we see that f is not a linear transformation.

(b) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sqrt{|x|}$.

Solution: Take $\mathbf{u} = 9$ and $\mathbf{v} = 16$. Then $\mathbf{u} + \mathbf{v} = 25$. We compute $f(\mathbf{u}) = 3$, $f(\mathbf{v}) = 4$ and $f(\mathbf{u} + \mathbf{v}) = 5$. Since

$$f(\mathbf{u} + \mathbf{v}) = 5 \neq 7 = f(\mathbf{u}) + f(\mathbf{v}),$$

we see that f is not a linear transformation.

(c) $f : \mathbb{R}^2 \to \mathbb{R}$ via $f(\mathbf{x}) = \|\mathbf{x}\|$ (length of vector $\mathbf{x}$).

Solution: Take $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We compute $f(\mathbf{u}) = 1$, $f(\mathbf{v}) = 1$ and $f(\mathbf{u} + \mathbf{v}) = \sqrt{2}$. Since

$$f(\mathbf{u} + \mathbf{v}) = \sqrt{2} \neq 2 = f(\mathbf{u}) + f(\mathbf{v}),$$

we see that f is not a linear transformation.

(d) $f : \mathbb{R}^3 \to \mathbb{R}^2$ via $f(x, y, z) = (2x + y + z, xyz)$.

SOLUTION: Take $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We compute $f(\mathbf{u}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $f(\mathbf{v}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $f(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Since

$$f(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 0 \end{bmatrix} = f(\mathbf{u}) + f(\mathbf{v}),$$

we see that f is not a linear transformation.

(e) $f : \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x, y) = (x + 1, y + 1)$.

SOLUTION: Take $\mathbf{u} = (0, 0)$ and $\mathbf{v} = (0, 0)$. Then $\mathbf{u} + \mathbf{v} = (0, 0)$. We compute $f(\mathbf{u}) = f(\mathbf{v}) = f(\mathbf{u} + \mathbf{v}) = (1, 1)$. Since

$$f(\mathbf{u} + \mathbf{v}) = (1, 1) \neq (2, 2) = f(\mathbf{u}) + f(\mathbf{v}),$$

we see that f is not a linear transformation.

2.) For each of the following linear transformations give the standard matrix for T:
(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ $(x, y) \mapsto (x + 2y, y)$.

SOLUTION: Apply T to $\mathbf{e}_1$ and $\mathbf{e}_2$ to get

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Then the standard matrix is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}$ via $T(x, y, z) = 3x - 2y + z$.

SOLUTION: Apply T to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to get

$$T(\mathbf{e}_1) = [3], \qquad T(\mathbf{e}_2) = [-2], \qquad T(\mathbf{e}_3) = [1].$$

Then the standard matrix is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; T(\mathbf{e}_3)] = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}.$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ via a clockwise rotation of 135° about the origin.

Solution: Apply T to $\mathbf{e}_1$ and $\mathbf{e}_2$ to get

$$T(\mathbf{e}_1) = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then the standard matrix is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2)] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ via reflection across the plane $x = z$ (i.e., $(x, y, z) \mapsto (z, y, x)$).

Solution: Apply T to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to get

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then the standard matrix is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; T(\mathbf{e}_3)] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ via rotation by 30° (clockwise, when viewed from vantage point $(0, 0, 10)$) about the x_3-axis.

Solution: Apply T to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to get

$$T(\mathbf{e}_1) = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \qquad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the standard matrix is

$$A = [T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; T(\mathbf{e}_3)] = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.) #24 on p81.

Solution: We must show that $T(\mathbf{x}) = \mathbf{0}$ for any vector $\mathbf{x}$ in $\mathbb{R}^n$. Okay, so let $\mathbf{x}$ be arbitrarily chosen. Since $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, we know we can find scalars $c_1, \ldots, c_p$ satisfying the vector equation

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p.$$

Let's apply the transfomation T to both sides of this equation:

$$T(\mathbf{x}) = T\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p\right).$$

Since T is a linear transformation, we can split up the right-hand side (see Eqn (5) on p77):

$$T(\mathbf{x}) = c_1T\left(\mathbf{v}_1\right) + c_2T\left(\mathbf{v}_2\right) + \cdots + c_pT\left(\mathbf{v}_p\right).$$

Since each $\mathbf{v}_i$ satisfies $T(\mathbf{v}_i) = \mathbf{0}$, this becomes

$$T(\mathbf{x}) = c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_p\mathbf{0},$$

or, simply,

$$T(\mathbf{x}) = \mathbf{0}$$

which is exactly what we wanted.

4.) Prove that the composition of two linear transformations is also a linear transformation: *If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are two linear transformations, then their composition*

$$S \circ T : \mathbb{R}^n \to \mathbb{R}^p$$

is a linear transformation.

Solution: We must show that $S \circ T$ satisfies the two key properties of a linear transformation. Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in $\mathbb{R}^n$. Then

$$\begin{aligned}
(S \circ T)(\mathbf{u} + \mathbf{v}) &= S\left(T(\mathbf{u} + \mathbf{v})\right) && \text{(Defn } S \circ T\text{)} \\
&= S\left(T(\mathbf{u}) + T(\mathbf{v})\right) && (T \text{ is a lin. trans.)} \\
&= S\left(T(\mathbf{u})\right) + S\left(T(\mathbf{v})\right) && (S \text{ is a lin. trans.)} \\
&= (S \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{v}) && \text{(Defn } S \circ T\text{)}
\end{aligned}$$

Likewise, if $\mathbf{u}$ is any vector and c is any scalar, we have

$$\begin{aligned}
(S \circ T)(c\mathbf{u}) &= S\left(T(c\mathbf{u})\right) && \text{(Defn } S \circ T\text{)} \\
&= S\left(cT(\mathbf{u})\right) && (T \text{ is a lin. trans.)} \\
&= cS\left(T(\mathbf{u})\right) && (S \text{ is a lin. trans.)} \\
&= c(S \circ T)(\mathbf{u}) && \text{(Defn } S \circ T\text{)}
\end{aligned}$$

Since these two properties hold for any $\mathbf{u}$, $\mathbf{v}$ and c, we now know that $S \circ T$ is a linear transformation.

5.) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation with the following property:

For any linearly independent vectors $\mathbf{v}_1$*,* $\mathbf{v}_2$ *and* $\mathbf{v}_3$ *in* $\mathbb{R}^3$*, the images* $T(\mathbf{v}_1)$*,* $T(\mathbf{v}_2)$ *and* $T(\mathbf{v}_3)$ *are linearly independent in* $\mathbb{R}^4$*.*

(a) Give an example of such a linear transformation. Give its standard matrix and the reduced row echelon form of this matrix.

(b) Work out all possible shapes of the reduced row echelon form for such a matrix. Use the symbols 0, 1, and $*$ where $*$ indicates an entry which may take on the value of any real number.

SOLUTION: For part (a), we can take $T(x, y, z) = (2x, 3y, 4z, 0)$, for example. The standard matrix for T is

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the reduced row echelon form has three pivots, we know that the images of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent. More generally, if

$$\begin{aligned} \mathbf{v}_1 &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{v}_2 &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\ \mathbf{v}_3 &= c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \end{aligned}$$

are any three linearly independent vectors in $\mathbb{R}^3$, then the matrix

$$[\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

has a pivot in every column. So then it is easy to see (by scaling rows) that the matrix

$$[T(\mathbf{v}_1)\,|T(\mathbf{v}_2)\,|T(\mathbf{v}_3)] = \begin{bmatrix} 2a_1 & 2b_1 & 2c_1 \\ 3a_2 & 3b_2 & 3c_2 \\ 4a_3 & 4b_3 & 4c_3 \\ 0 & 0 & 0 \end{bmatrix}$$

will also have a pivot in every column. So the images of these three vectors will be linearly independent in $\mathbb{R}^4$ as required.

For part (b), we show that the above rref is the ONLY possible rref of any standard matrix of a linear transformation T with the given property. Indeed, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are linearly independent. So the problem requires their images to also be independent. But these are the columns of A. So the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Therefore the rref of A must have a pivot in every column. But in the rref, every entry above and below a pivot must be zero. So the rref of A is exactly the matrix above: a 3×3 identity matrix with a row of zeros added onto the bottom.