Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Assignment 3

DUE DATE: Tuesday, February 2, 4:30pm. Place in mail bin labelled "MA2071" in Room SH108. (The label is red and the bin is near the bottom row, underneath the staff mail slots.)
PLEASE WRITE YOUR NAME AND SECTION NUMBER ON YOUR ASSIGNMENT.

Please recall Professor Martin's rules for Linear Algebra assignments, which are reproduced at the end of this assignment for your convenience.

Please complete the following five problems.

1. For each of the following functions f from $\mathbb{R}^n$ to $\mathbb{R}^m$, show that f is **not** a linear transformation. Be specific! Give explicit vectors and constants (if necessary) which show a part of the definition which fails for this particular function f.
   (a) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = -5$.
   (b) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sqrt{|x|}$.
   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ via $f(\mathbf{x}) = \|\mathbf{x}\|$ (length of vector $\mathbf{x}$).
   (d) $f : \mathbb{R}^3 \to \mathbb{R}^2$ via $f(x, y, z) = (2x + y + z, xyz)$.
   (e) $f : \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x, y) = (x + 1, y + 1)$.

2. For each of the following linear transformations, described in words, give the standard matrix for T:
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ $(x, y) \mapsto (x+2y, y)$ (i.e., the function adds twice the second coordinate to the first and leaves the second coordinate unchanged).
   (b) $T : \mathbb{R}^3 \to \mathbb{R}$ via $T(x, y, z) = 3x - 2y + z$.
   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ via a clockwise rotation of $135°$ about the origin.
   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ via reflection across the plane $x = z$ (i.e., $(x, y, z) \mapsto (z, y, x)$).
   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ via rotation by $30°$ (clockwise, when viewed from vantage point $(0, 0, 10)$) about the x_3-axis ($x_1 = x_2 = 0$ is the equation for this line, by the way).
   IMPORTANT: Show the process by which you obtain each matrix.

3. #24 on p81.

4. Prove that the composition of two linear transformations is also a linear transformation:

   If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are two linear transformations, then their composition

   $$S \circ T : \mathbb{R}^n \to \mathbb{R}^p$$

   is a linear transformation.

5. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation with the following property:

   > *For any linearly independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in $\mathbb{R}^3$, the images $T(\mathbf{v}_1)$, $T(\mathbf{v}_2)$ and $T(\mathbf{v}_3)$ are linearly independent in $\mathbb{R}^4$.*

   (a) Give an example of such a linear transformation. Give its standard matrix and the reduced row echelon form of this matrix.

   (b) Work out all possible shapes of the reduced row echelon form for such a matrix. Use the symbols 0, 1, and $*$ where $*$ indicates an entry which may take on the value of any real number. (Cf. Example 1 on p15.[1])

PROF. MARTIN'S RULES FOR LINEAR ALGEBRA ASSIGNMENTS:
I) Each student must compose his/her assignments independently. However, brainstorming may be done in groups;
II) Use only one side of each sheet of paper. Write legibly using correct English;
III) Show your work. Explain your answers using FULL SENTENCES;
IV) Use a staple when you submit more than one sheet and want them all back. There is a stapler for public use in the Mathematical Sciences Department Office (SH108) where assignments are turned in;
V) No late assignments will be accepted for credit.

[1] For example, both the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ are special cases of the shape $\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$ but neither one is in reduced row echelon form.