Linear Algebra
C Term, Sections C01-C04
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Sample Solutions
Linear Algebra Assignment 2

Problem 1:
(a) We must determine whether or not $\mathbf{w}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$) where

$$\mathbf{w} = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 2 \\ 2 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ -3 \\ -1 \end{bmatrix}.$$

Solution: To find all solutions to the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w},$$

we row reduce the augmented matrix $[A|w] = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{w}]$:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 1 & 5 & 4 & 3 \\ 1 & 2 & -3 & 5 \\ 1 & 2 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -6 \end{array}\right] \begin{array}{l} (R1) \\ (R2)-(R1) \\ (R3)-(R1) \\ (R4)-(R1) \end{array}.$$

We can stop right here: there is a row with all zeros on the left and a non-zero value in the last column. By Theorem 2, the system is inconsistent. There is no solution. Therefore
Conclusion: NO, $\mathbf{w}$ is not in the span of these vectors. □

(b) Find a simple equation involving the entries of $\mathbf{w}$ that guarantees that $\mathbf{w} = (w_1, w_2, w_3)$ is in the span of

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

Solution: Just as with part (a), we row reduce the augmented matrix $[A|w] = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{w}]$:

$$\left[\begin{array}{rrr|r} 2 & 4 & 0 & w_1 \\ 2 & 0 & 8 & w_2 \\ 1 & -4 & -4 & w_3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 2 & 4 & 0 & w_1 \\ 0 & -4 & 8 & w_2 - w_1 \\ 0 & -6 & -4 & w_3 - \frac{1}{2}w_1 \end{array}\right] \begin{array}{l} (R1) \\ (R2)-(R1) \\ (R3)-\frac{1}{2}(R1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & 0 & w_1 \\ 0 & -4 & 8 & w_2 - w_1 \\ 0 & 0 & -16 & w_1 + w_3 - \frac{3}{2}w_2 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3) - \frac{3}{2}(R2) \end{array}$$

This is strange! The problem said to decide when **w** belongs to the span. But now we have a pivot in every row on the left-hand side. So **every** vector **w** is in the span: the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span all of $\mathbb{R}^3$. So an example of an equation that must hold true in order for **w** to be in the span is

$$0 = 0, \qquad \text{or} \qquad w_1 = w_1. \square$$

VARIATION: For the purpose of exam preparation, let's consider a slightly different problem where the zero is replaced by a 16:

Find a simple equation involving the entries of **w** that guarantees that $\mathbf{w} = (w_1, w_2, w_3)$ is in the span of

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ 16 \\ -4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

Solution: Again, we row reduce the augmented matrix $[A|w] = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{w}]$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & w_1 \\ 2 & 16 & 8 & w_2 \\ 1 & -4 & -4 & w_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 0 & w_1 \\ 0 & 12 & 8 & w_2 - w_1 \\ 0 & -6 & -4 & w_3 - \frac{1}{2}w_1 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - (R1) \\ (R3) - \frac{1}{2}(R1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 4 & 0 & w_1 \\ 0 & 12 & 8 & w_2 - w_1 \\ 0 & 0 & 0 & w_3 - w_1 + \frac{1}{2}w_2 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3) + \frac{1}{2}(R2) \end{array}$$

We can now see two possibilities for the echelon here: If

$$w_2 = 2w_1 - 2w_3,$$

then we will get a row of all zeros and **w** is in the span. But if $w_2 \neq 2w_1 - 2w_3$, then the system is inconsistent (Theorem 2) and **w** is not in the span of these vectors. $\square$

Problem 2:

(a) Suppose you know that the linear system

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 3x_3 & + & 15x_4 & + & x_5 & = & -1 \\ 2x_1 & + & x_2 & - & x_3 & - & 3x_4 & & & = & 5 \\ -x_1 & - & x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & -15 \\ 4x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & + & 2x_5 & = & 3 \end{array}$$

has solution set

$$\left\{ \begin{array}{rcrcrcr} x_1 & = & 2 & & & + & 4s \\ x_2 & = & 1 & + & r & - & 5s \\ x_3 & = & & & r & & \\ x_4 & = & & & & & s \\ x_5 & = & -4 & & & & \end{array} \right. \qquad r, s \in \mathbb{R}$$

Write down the solution set to the linear system

$$\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 3x_3 & + & 15x_4 & + & x_5 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & - & 3x_4 & & & = & 0 \\ -x_1 & - & x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & 0 \\ 4x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & + & 2x_5 & = & 0 \end{array}$$

Solution: We apply Theorem 6 here. To get all solutions to the homogeneous system, we simply subtract off the particular solution $\mathbf{p} = (2, 1, 0, 0, -4)$ from each solution to the system $A\mathbf{x} = \mathbf{b}$: our new solution set is

$$\left\{ \begin{array}{rcrcr} x_1 & = & & & 4s \\ x_2 & = & r & - & 5s \\ x_3 & = & r & & \\ x_4 & = & & & s \\ x_5 & = & & & \end{array} \right. \qquad (r, s \in \mathbb{R})$$

(b) Suppose you know that the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 0 \\ x_1 & + & 3x_2 & + & 2x_3 & & & = & 0 \\ x_1 & + & 3x_2 & & & - & 2x_4 & = & 0 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 3x_4 & = & 0 \end{array}$$

has solution set

$$\left\{ \begin{array}{rcrcr} x_1 & = & -3r & + & 2s \\ x_2 & = & r & & \\ x_3 & = & & - & s \\ x_4 & = & & & s \end{array} \right. \qquad r, s \in \mathbb{R}$$

Without any further computation, write down the solution set to the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 6 \\ x_1 & + & 3x_2 & + & 2x_3 & & & = & 6 \\ x_1 & + & 3x_2 & & & - & 2x_4 & = & 6 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 3x_4 & = & 12 \end{array}$$

Solution: We again apply Theorem 6, but this time we have a bit of work to do: we need to find just one solution $\mathbf{p}$ to the system

Observe that $\mathbf{p} = (0, 2, 0, 0)$ is a solution since

$$A\mathbf{p} = 2 \begin{bmatrix} 3 \\ 3 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 12 \end{bmatrix}.$$

So, to get all solutions to the non-homogeneous system, we simply add the particular solution $\mathbf{p} = (0, 2, 0, 0)$ to each solution to the homogeneous system: our new solution set is

$$\left\{ \begin{array}{rcl} x_1 & = & -3r + 2s \\ x_2 & = & 2 + r \\ x_3 & = & -s \\ x_4 & = & s \end{array} \right. \qquad (r, s \in \mathbb{R})$$

Problem 3: Find all values of h and k for which the columns of A span $\mathbb{R}^n$.

(a) $n = 2, \qquad A = \begin{bmatrix} 2 & -3 \\ h & k \end{bmatrix}$

Solution: We apply Theorem 4 which says that the columns of A span $\mathbb{R}^n$ precisely when A has a pivot in every row. In this case, we row reduce

$$A = \begin{bmatrix} 2 & -3 \\ h & k \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & k + \frac{3}{2}h \end{bmatrix} \begin{array}{l} (R1) \\ (R2) - \frac{3}{2}(R1) \end{array}$$

and see that there will be a pivot in the second row if and only if $3h + 2k \neq 0$.

(b) $n = 3, \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & h & 4 \\ 0 & 0 & k \end{bmatrix}$

Solution: Again, we apply Theorem 4. In this case, if h and k are both nonzero, we are already in row echelon form and there is a pivot in every row. If $h = 0$, then we get no pivot in column two and then we won't have a pivot in row 3 when we're through. So that stinks. Likewise, if $k = 0$, then we have a row of zeros at the bottom and there is no way we'll get a pivot in th third row. So the columns can't span $\mathbb{R}^3$ in that case either.
Conclusion: The columns span if and only if both h and k are not zero.

(c) $n = 3, \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & k \\ -1 & -h & h-k \end{bmatrix}$

Solution: Same strategy again:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & k \\ -1 & -h & h-k \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & h-1 & k-1 \\ 0 & 1-h & 1+h-k \end{bmatrix} \begin{array}{l} (R1) \\ (R2) - (R1) \\ (R3) + (R1) \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & h-1 & k-1 \\ 0 & 0 & h \end{bmatrix} \begin{array}{l} (R1) \\ (R2) \\ (R3) + (R2) \end{array}$$

and we can now answer the question using Theorem 4:
CONCLUSION: The columns span if and only if both $h \neq 1$ and $n \neq 0$. The value of k is irrelevant.

Problem 4: Exercise #26 on page 56 in the text.

Solution: Assume first that $A\mathbf{x} = \mathbf{b}$ is a consistent linear system and let $\mathbf{p}$ be such a solution. Now suppose the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then, by Theorem 6, every solution to $A\mathbf{x} = \mathbf{b}$ is of the form $\mathbf{p} + \mathbf{v}_h$ for some solution $\mathbf{v}_h$ to the homogeneous system. Since the only possibility is $\mathbf{v}_h = \mathbf{0}$, the only solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{p} + \mathbf{0} = \mathbf{p}$. So the solution is unique.

Now the same ideas give us the other direction. Assume that $A\mathbf{x} = \mathbf{b}$ has only one solution $\mathbf{p}$. Since every solution $\mathbf{v}_h$ to $A\mathbf{x} = \mathbf{0}$ gives us a solution $\mathbf{b} + \mathbf{v}_h$ to $A\mathbf{x} = \mathbf{b}$ and two different $\mathbf{v}_h$ vectors would give two different solutions, we must conclude that there is only one possibility for $\mathbf{v}_h$, namely the trivial solution which, we know, is always present for a homgeneous system. □

Problem 5: We need to find eigenvectors.

(a) $A = \begin{bmatrix} -2 & -1 \\ 5 & -8 \end{bmatrix}, \qquad \lambda = -3$

Solution: We row reduce the augmented matrix $[A - \lambda I | \mathbf{0}] =$

$$\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - 5(R1) \end{array}$$

and find all eigenvectors

$$\begin{cases} x_1 = r \\ x_2 = r \end{cases} \qquad (r \in \mathbb{R},\ r \neq 0)$$

(b) $A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 4 \\ -3 & 3 & -5 \end{bmatrix}, \qquad \lambda = 1$

Solution: We row reduce the augmented matrix $[A - \lambda I | \mathbf{0}] =$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 0 \\ 2 & -2 & 4 & 0 \\ -3 & 3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - 2(R1) \\ (R3) + 3(R1) \end{array}$$

and find all eigenvectors

$$\begin{cases} x_1 &= r \quad -2s \\ x_2 &= r \\ x_3 &= \qquad s \end{cases} \qquad (r, s \in \mathbb{R}, \quad \text{not both zero})$$

(c) $A = \begin{bmatrix} 6 & 0 & -2 & 0 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 5 & -3 \\ -3 & 0 & 6 & 5 \end{bmatrix}, \qquad \lambda = 5$

Solution: We again subtract λ off each diagonal entry of A and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ -3 & 0 & 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - (R1) \\ (R3) \\ (R4) + 3(R1) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ (R3) \\ (R2) + (R3) \\ \end{array} \qquad \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) + (R3) \\ \\ \frac{1}{2}(R3) \\ \end{array}$$

Now we see that there is one free parameter and the eigenvectors are

$$\begin{cases} x_1 &= 4r \\ x_2 &= 3r \\ x_3 &= 2r \\ x_4 &= r \end{cases} \qquad (r \neq 0).$$