Linear Algebra
C Term, Sections C01-C04
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Linear Algebra Assignment 2

DUE DATE: Tuesday, January 26, 4:30pm. Place in mail bin labelled "MA2071" in Room SH108. (The label is red and the bin is near the bottom row, underneath the staff mail slots.)
ALWAYS WRITE BOTH YOUR NAME AND YOUR SECTION NUMBER ON YOUR ASSIGNMENT.

Please recall Professor Martin's rules for Linear Algebra assignments, which are reproduced at the end of this assignment for your convenience.

Please complete the following five problems.

1. **(a)** Determine whether or not $\mathbf{w}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$) where

$$\mathbf{w} = \begin{bmatrix} 3 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ -3 \\ -1 \end{bmatrix}.$$

(b) Find a simple equation involving the entries of $\mathbf{w}$ that guarantees that $\mathbf{w} = (w_1, w_2, w_3)$ is in the span of

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

2. **(a)** Suppose you know that the linear system

$$\begin{array}{rrrrrrrrrrr} & & 3x_2 & - & 3x_3 & + & 15x_4 & + & x_5 & = & -1 \\ 2x_1 & + & x_2 & - & x_3 & - & 3x_4 & & & = & 5 \\ -x_1 & - & x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & -15 \\ 4x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & + & 2x_5 & = & 3 \end{array}$$

has solution set

$$\begin{cases} x_1 = 2 + 4s \\ x_2 = 1 + r - 5s \\ x_3 = r \\ x_4 = s \\ x_5 = -4 \end{cases} \qquad r, s \in \mathbb{R}$$

Without any further computation, write down the solution set to the linear system

$$\begin{array}{rcrcrcrcrcl} & & 3x_2 & - & 3x_3 & + & 15x_4 & + & x_5 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & - & 3x_4 & & & = & 0 \\ -x_1 & - & x_2 & + & x_3 & - & x_4 & + & 3x_5 & = & 0 \\ 4x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & + & 2x_5 & = & 0 \end{array}$$

Explain!

(b) Suppose you know that the linear system

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 0 \\ x_1 & + & 3x_2 & + & 2x_3 & & & = & 0 \\ x_1 & + & 3x_2 & & & - & 2x_4 & = & 0 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 3x_4 & = & 0 \end{array}$$

has solution set

$$\begin{cases} x_1 = -3r + 2s \\ x_2 = r \\ x_3 = -s \\ x_4 = s \end{cases} \qquad r, s \in \mathbb{R}$$

Without any further computation, write down the solution set to the linear system

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & - & x_3 & - & 3x_4 & = & 6 \\ x_1 & + & 3x_2 & + & 2x_3 & & & = & 6 \\ x_1 & + & 3x_2 & & & - & 2x_4 & = & 6 \\ 2x_1 & + & 6x_2 & + & x_3 & - & 3x_4 & = & 12 \end{array}$$

Explain! (HINT: Look at the coefficients of x_2.)

3. In each of the following, find all values of h and k for which the columns of A span $\mathbb{R}^n$. Explain.

(a) $n = 2, \qquad A = \begin{bmatrix} 2 & -3 \\ h & k \end{bmatrix}$

(b) $n = 3,$ $$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & h & 4 \\ 0 & 0 & k \end{bmatrix}$$

(c) $n = 3,$ $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & k \\ -1 & -h & h-k \end{bmatrix}$$

4. Exercise #26 on page 56 in the text. (The correct solution will be in paragraph form, consisting of two parts, one for each of the two logical implications in the second sentence.)

First, some background for Problem 5:
If A is a square $n \times n$ matrix and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, we say $\mathbf{x}$ is an *eigenvector* for A if $A\mathbf{x}$ is equal to a scalar multiple of $\mathbf{x}$: $A\mathbf{x} = \lambda\mathbf{x}$ for some real number λ (called the *eigenvalue*).

Given a matrix A and an eigenvalue λ for A (we will learn later how to find these special λ), the process of finding all eigenvectors of A corresponding to λ reduces to our familiar row reduction algorithm, as follows.

- Suppose $A\mathbf{x} = \lambda\mathbf{x}$
- Recall the $n \times n$ identity matrix I is the matrix with ones down the main diagonal and zeros elsewhere. Since $I\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$, we can write the above equation as
$$A\mathbf{x} = \lambda(I\mathbf{x})$$
- Now we use the algebraic rules on pages 32 and 45 in the text to get
$$\begin{aligned} A\mathbf{x} &= \lambda(I\mathbf{x}) \\ A\mathbf{x} &= (\lambda I)\mathbf{x} \\ A\mathbf{x} - (\lambda I)\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\mathbf{x} &= \mathbf{0} \end{aligned}$$
- So we need only to compute the matrix $A - \lambda I$ (this is easy: just subtract the constant λ off each entry along the main diagonal of A) and find all solutions to the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. Any non-zero vector $\mathbf{x}$ which solves this system is an eigenvector corresponding to λ. (Note that, for eigenvector problems, we always get right-hand-side values all equal to zero.)

Now your homework problem:

5. Find all eigenvectors (in the same form as (5) on p21) for the following problems:

(a) $A = \begin{bmatrix} -2 & -1 \\ 5 & -8 \end{bmatrix}, \qquad \lambda = -3$

(b) $A = \begin{bmatrix} 2 & -1 & 2 \\ 2 & -1 & 4 \\ -3 & 3 & -5 \end{bmatrix}, \qquad \lambda = 1$

(c) $A = \begin{bmatrix} 6 & 0 & -2 & 0 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 5 & -3 \\ -3 & 0 & 6 & 5 \end{bmatrix}, \qquad \lambda = 5$

Prof. Martin's Rules for Linear Algebra Assignments:

I) Each student must compose his/her assignments independently. However, brainstorming may be done in groups;

II) **Use only one side of each sheet of paper**. Write legibly using correct English;

III) Show your work. Explain your answers using FULL SENTENCES;

IV) Use a staple when you submit more than one sheet and want them all back. There is a stapler for public use in the Mathematical Sciences Department Office (SH108) where assignments are turned in;

V) **No late assignments will be accepted for credit.**