Linear Algebra
C Term, Sections C01-C04
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Sample Solutions

Linear Algebra Assignment 1

Problem 1: We have the linear system:

$$\begin{array}{rrrrrcr} 2x_1 & -2x_2 & +2x_3 & & -2x_5 & = & 16 \\ x_1 & +\ x_2 & +5x_3 & & +9x_5 & = & 8 \\ -x_1 & & -3x_3 & +x_4 & +2x_5 & = & -1 \\ x_1 & & +3x_3 & & +4x_5 & = & 8 \end{array}$$

(a) Write down the augmented matrix corresponding to this system.
Solution:

$$[A|\mathbf{b}] = \left[\begin{array}{rrrrr|r} 2 & -2 & 2 & 0 & -2 & 16 \\ 1 & 1 & 5 & 0 & 9 & 8 \\ -1 & 0 & -3 & 1 & 2 & -1 \\ 1 & 0 & 3 & 0 & 4 & 8 \end{array}\right]$$

(b) Use the row reduction algorithm on this matrix to obtain a row equivalent matrix in **reduced row echelon form**.
Solution:

$$[A|\mathbf{b}] \sim \left[\begin{array}{rrrrr|r} 1 & 0 & 3 & 0 & 4 & 8 \\ 0 & 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 7 \\ 0 & -2 & -4 & 0 & -10 & 0 \end{array}\right] \begin{array}{l} (R4) \\ (R2)-(R4) \\ (R3)+(R4) \\ (R1)-2(R4) \end{array}$$

$$\sim \left[\begin{array}{rrrrr|r} 1 & 0 & 3 & 0 & 4 & 8 \\ 0 & 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} (R1) \\ (R2) \\ (R3) \\ (R4)+2(R2) \end{array}$$

which is in reduced row echelon form.
(c) Using part (b), find all solutions to the original linear system. Describe the solution set in parametric form as in (5) and (7) on p21-22 in the text.
Solution: Here are all the solutions to the original system:

$$\begin{array}{rclrrl} x_1 & = & 8 & -\ 3r & -\ 4s & \\ x_2 & = & & -\ 2r & -\ 5s & \\ x_3 & = & & r & & (free) \\ x_4 & = & 7 & & -\ 6s & \\ x_5 & = & & & s & (free) \end{array} \qquad \mathbf{x} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 7 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ -5 \\ 0 \\ -6 \\ 1 \end{bmatrix} \qquad (r, s \in \mathbb{R})$$

(The form on the left is the one required, but I also give it in "parametric vector form" on the right.)

Problem 2: We are given a linear system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & k \\ 2 & h & 7 & 10 \\ 3 & 3 & 15 & 12 \end{array}\right]$$

and are asked which values of h and k will give (a) an inconsistent system; (b) a system with a unique solution; (c) a system with infinitely many solutions.
Solution: To decide these questions, we need only row reduce to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & k \\ 2 & h & 7 & 10 \\ 3 & 3 & 15 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 5 & k \\ 0 & h-2 & -3 & 10-2k \\ 0 & 0 & 0 & 12-3k \end{array}\right] \begin{array}{l} (R1) \\ (R2)-2(R1) \\ (R3)-3(R1) \end{array}$$

We must stop here and consider the values of h and k.
case (i): If $k \neq 4$, then we get a pivot in the last column and the system is inconsistent.

case (ii): Now let's assume $k = 4$. Then we get a row of all zeros and we can ignore it. If $h \neq 2$, then we are already in row echelon form and x_3 is a free variable. So we have infinitely many solutions. But if $h = 2$, then we are still in row echelon form with a pivot in position (2,3) instead of (2,2). So x_2 would be a free variable in that case and we again get hoardes of solutions.
SUMMARY:
(a) The system is inconsistent for $k \neq 4$ and any value of h.
(b) It never occurs that such a system has a unique solution.
(a) The system has infinitely many solutions for $k = 4$ and any value of h.

Problem 3: Exercise 12 on p64.
Solution: The network we analyze is shown on p64 of the text. The various traffic flows are governed by the linear system

Flow Conservation at A: $x_1 \quad - x_3 - x_4 \quad = 40$
Flow Conservation at B: $x_1 + x_2 \quad = 200$
Flow Conservation at C: $x_2 + x_3 \quad - x_5 = 100$
Flow Conservation at D: $x_4 + x_5 = 60$

(a) To find the general solution, we construct the augmented matrix for the above linear system and row reduce.

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right] (R2)-(R1)$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{array}\right] \quad (R3)-(R2) \qquad \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad (R4)+(R3)$$

We have reached echelon form. Now we proceed backwards to reduced echelon form.

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} (R1)-(R3) \\ (R2)+(R3) \\ -(R3) \end{array}$$

Now we have reached r.r.e.f. All possible solutions are easily read off from this final matrix: the pivot columns are 1, 2 and 4. So x_1, x_2 and x_4 are basic variables while x_3 and x_5 are free variables. I like to use parameters like r and s for the free variables; the solutions are

$x_1 = 100 + r - s$
$x_2 = 100 - r + s$
$x_3 = r$ (any real number)
$x_4 = 60 - s$
$x_5 = s$ (any real number)
This is a full description of the general traffic pattern in the network.

(b) When the road from Point A to Point D is closed. We have $x_4 = 0$. So we must restrict the above solution by setting $s = 60$. We now have one free parameter:

$x_1 = 40 + r$
$x_2 = 160 - r$
$x_3 = r$ (any real number)
$x_4 = 0$
$x_5 = 60$

(c) Up until now, we've ignored one feature of the real-world problem not captured by linear algebra: all traffic flows must be non-negative. So x_1 cannot be set equal to *any* real number; it is restricted by the conditions $x_2 \geq 0$, $x_3 \geq 0$. These give $0 \leq r \leq 160$ which, in turn, give the bounds $40 \leq x_1 \leq 200$. So the minimum value of x_1 in reality is 40 cars/minute.

Problem 4:

(a) Determine all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ which satisfy $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution: We have

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{bmatrix}.$$

So the matrix equation $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives us four equations:

$$\begin{aligned} a^2 + bc &= 0 && (1) \\ b(a+d) &= 0 && (2) \\ c(a+d) &= 0 && (3) \\ d^2 + bc &= 0 && (4) \end{aligned}$$

We consider two cases.

case (i): Assume $b = 0$. Then the first equation gives $a = 0$, the last gives $d = 0$ and that's all we get. So this gives us an infinite collection of matrices of the form $A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$, where c can be any real number, satisfying the equation $A^2 = 0$.

case (ii): Any other solution must have $b \neq 0$, so assume that. We can then cancel the b in Eqn (2) and find that $d = -a$. Making that substitution, the system boils down to

$$a^2 + bc = 0, \qquad d = -a, \qquad b \neq 0.$$

So we can divide by b and solve for c: $c = -a^2/b$. This gives us infinitely many matrices

$$A = \begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix}$$

where a can be any real number and b can be any nonzero number. (For example, with $a = 15$ and $b = -9$, we find $A = \begin{bmatrix} 15 & -9 \\ 25 & -15 \end{bmatrix}$, which is easily seen to satisfy $A^2 = 0$.

Problem 5: By computing the inner product $\langle f, g \rangle := \int_0^1 f(x)g(x)\,dx$ for each of the six possible pairs, we determine which pairs among the following functions are orthogonal pairs:
(a) $a(x) = 1$ for $0 \leq x \leq 1$
(b) $b(x) = x$ for $0 \leq x \leq 1$
(c) $c(x) = x - \frac{1}{2}$ for $0 \leq x \leq 1$
(d) $d(x) = 6x^2 - 6x + 1$ for $0 \leq x \leq 1$
Solution: We compute

$$\langle a, b \rangle = \int_0^1 x\,dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}.$$

So, a and b **are not** orthogonal.

Next, we decide if vectors a and c are orthogonal:

$$\langle a, c \rangle = \int_0^1 x - \frac{1}{2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}x \Big|_0^1 = 0.$$

So, $a \perp c$: they **are** orthogonal.

Likewise, we consider a and d:

$$\langle a, d\rangle = \int_0^1 6x^2 - 6x + 1\,dx = 2x^3 - 3x^2 + x\Big|_0^1 = 0.$$

So, $a \perp d$: they **are** orthogonal.

Moving along, we next consider vectors b and c:

$$\langle b, c\rangle = \int_0^1 x^2 - \frac{1}{2}x\,dx = \frac{1}{3}x^3 - \frac{1}{4}x^2\Big|_0^1 = \frac{1}{12}.$$

So, b and c **are not** orthogonal.

Now consider b and d:

$$\langle b, d\rangle = \int_0^1 6x^3 - 6x^2 + x\,dx = \frac{3}{2}x^4 - 2x^3 + \frac{1}{2}x^2\Big|_0^1 = 0.$$

So, $b \perp d$: they **are** orthogonal.

The hardest one is the last one. The "dot product" of c and d is

$$\langle c, d\rangle = \int_0^1 \left(6x^3 - 9x^2 + 4x - \frac{1}{2}\right)\,dx = \frac{3}{2}x^4 - 3x^3 + 2x^2 - \frac{1}{2}x\Big|_0^1 = 0.$$

So, $c \perp d$: these two **are** also orthogonal.