C Term, Sections C01-C04 Instructor: W. J. Martin
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Matrices and Linear Algebra I – Test 3

Sample Solutions

1.) For each of the following, determine whether or not W is a subspace of the vector space M_{22}. If W **is** a subspace, prove that both properties hold. If W is **not** a subspace, show that some property fails using concrete examples.

(a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; a+d = b+c \right\}$

Solution: **YES** this is a subspace. First, suppose $\mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ belong to W. Then $a+d = b+c$ and $a'+d' = b'+c'$ giving us $(a+a')+(d+d') = (b+b')+(c+c')$ which shows that $\mathbf{u}+\mathbf{v} = \begin{bmatrix} a+a' & b+b' \\ c+c' & a+d' \end{bmatrix}$ belongs to W. Similarly, if r is any real number, then $r \cdot \mathbf{u} \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$ lies in W since $r^2ad = r^2bc$.

(b) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; a \leq b \right\}$

Solution: **NO** this is not a subspace. For example, $\mathbf{u} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ lies in W since $a = 0$ and $b = 1$ here. But $(-1) \cdot \mathbf{u} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ does not belong to W.

(c) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; ad - bc = 0 \right\}$

Solution: **NO** this is not a subspace. For example, both $\mathbf{u} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ belong to W, but $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 & 4 \\ 3 & 9 \end{bmatrix}$ does not since $1 \cdot 9 \neq 3 \cdot 4$. So W is not closed under addition.

(d) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; : \; b = 0 \right\}$

Solution: **YES** this is a subspace. First, suppose

$$\mathbf{u} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} a' & 0 \\ c' & d' \end{bmatrix}$$

belong to W. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a + a' & 0 \\ c + c' & a + d' \end{bmatrix}$$

belongs to W. Similarly, if r is any real number, then

$$r \cdot \mathbf{u} \begin{bmatrix} ra & 0 \\ rc & rd \end{bmatrix}$$

lies in W.

2.) Determine which of the following are bases for the vector space P_3.

(a) $S_1 = \{t^3 + t + 1,\ t + 1,\ t^3 + t^2 + t + 1,\ t^2 + t,\ t^3 + 1\}$

Solution: **NO** this is not a basis. The space P_3 has dimension four and S_1 has 5 vectors.

(b) $S_2 = \{t^3 + 3t^2 + t + 2,\ t^2 + 2t + 3,\ 3t + 6,\ 5\}$

Solution: **YES** this is a basis. It consists of $\dim(P_3)$ vectors and they are linearly independent. Indeed, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$, then

$$\begin{array}{rcrcrcrcl} c_1 & & & & & & & = & 0 \\ 3\ c_1 & + & c_2 & & & & & = & 0 \\ c_1 & + & 2\ c_2 & + & 3\ c_3 & & & = & 0 \\ 2\ c_1 & + & 3\ c_2 & + & 6\ c_3 & + & 5\ c_4 & = & 0 \end{array}$$

giving $c_1 = 0$ (from the 1st eqn) which forces $c_2 = 0$ (2nd eqn) then $c_3 = 0$ and $c_4 = 0$ in turn.

(c) $S_3 = \{t^3 + t + 2,\ t^2 + 2t + 3,\ 3t^3 + 3t + 6,\ t^3 + 1\}$

Solution: **NO**, this is not a basis. We observe that $\mathbf{v}_3$ is dependent on $\mathbf{v}_1$. Explicitly,

$$-3(t^3 + t + 2) + (3t^3 + 3t + 6) = 0t^3 + 0t^2 + 0t + 0$$

showing that S_2 is linearly dependent.

(d) $S_4 = \{t^3 + t^2 + 2t - 1,\ t^2 + 3,\ -2t^3 - 4t,\ 3t^3 + 5t^2 + 6t\}$

Solution: **NO**, this is not a basis. One can easily see that S_4 does not span. For if

$$at^3 + bt^2 + ct + d = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

is any polynomial in $\mathsf{span}(S_4)$, then $c = 2a$ since this relationship holds for each $\mathbf{v}_i$.

3.) In parts (a)-(c), we study the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & 2 & -1 & 0 \\ 2 & -1 & -2 & -2 & 3 & -5 \\ 1 & 1 & -1 & 2 & 1 & -2 \\ 4 & -2 & -4 & -4 & 0 & 14 \end{bmatrix}.$$

For your convenience, here is the (unique) matrix in reduced row echelon form which is row equivalent to A:

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the row space of A:

Solution:

$$S = \{[1,0,-1,0,0,3],\ [0,1,0,2,0,-1],\ [0,0,0,0,1,-4]\}$$

These are the non-zero rows of the reduced row echelon form.

(b) Find a basis for the column space of A:

Solution: We take the columns of the original matrix where pivots occur in the r.r.e.f.:

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) Find a basis for the nullspace of A:

Solution: The solutions to the linear system are all vectors of the form

$$[\, r - 3t, \quad t - 2s, \quad r, \quad s, \quad 4t, \quad t \,]$$

where r, s and t can be any real numbers. So we obtain three basis vectors:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$$

4.) Find a subset of S that is a basis for $W = \mathsf{span}(S)$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 5 \\ 2 \end{bmatrix} \right\}$$

Solution: To do this, we build a matrix with the given vectors $\mathbf{v}_j$ as its columns:

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ -3 & 6 & 0 & -5 & 5 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}.$$

Now we quickly row reduce this matrix to obtain

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(We can stop here.) Since the pivots are in columns 1, 3 and 4, we take as our basis

$$S = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -5 \\ 0 \end{bmatrix} \right\}.$$