C Term, Sections C01-C04 Instructor: W. J. Martin
Date: February 24, 2003 Type a

Matrices and Linear Algebra I – Test 3
Sample Solutions

1.) In parts (a)-(c), we study the matrix

$$A = \begin{bmatrix} 2 & -1 & -2 & 1 & 3 & -4 \\ -1 & 1 & 1 & -1 & -1 & 0 \\ 4 & -2 & -4 & 2 & 0 & 10 \\ 1 & 1 & -1 & -1 & 1 & -2 \end{bmatrix}.$$

For your convenience, here is the (unique) matrix in reduced row echelon form which is row equivalent to A:

$$R = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the row space of A:

Solution:

$$S = \{[1,0,-1,0,0,2],\ [0,1,0,-1,0,-1],\ [0,0,0,0,1,3]\}$$

These are the non-zero rows of the reduced row echelon form.

(b) Find a basis for the column space of A:

Solution: We take the columns of the original matrix where pivots occur in the r.r.e.f.:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) Find a basis for the nullspace of A:

Solution: The solutions to the linear system are all vectors of the form

$$[\ r-2t,\quad s+t,\quad r,\quad s,\quad 3t,\quad t\ ]$$

where r, s and t can be any real numbers. So we obtain three basis vectors:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

2.) Determine which of the following are bases for the vector space M_{22}.

(a) $S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Solution: **NO** this is not a basis. The space M_{22} has dimension four and S_1 has 5 vectors.

(b) $S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Solution: **NO**, this is not a basis. We observe that $\mathbf{v}_3$ is dependent on $\mathbf{v}_1$. Explicitly,

$$-2\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

showing that S_2 is linearly dependent.

(c) $S_3 = \left\{ \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 6 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \right\}$

Solution: **YES** this is a basis. It consists of $\dim(M_{22})$ vectors and they are linearly independent. Indeed, if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$, then

$$\begin{array}{rcrcrcrcl} c_1 & & & & & & & = & 0 \\ 4c_1 & + & c_2 & & & & & = & 0 \\ c_1 & + & 2c_2 & + & 3c_3 & & & = & 0 \\ 2c_1 & + & 3c_2 & + & 6c_3 & + & 3c_4 & = & 0 \end{array}$$

giving $c_1 = 0$ (from the 1st eqn) which forces $c_2 = 0$ (2nd eqn) then $c_3 = 0$ and $c_4 = 0$ in turn.

(d) $S_4 = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ -4 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 6 & 0 \end{bmatrix} \right\}$

Solution: **NO**, this is not a basis. One can easily see that S_4 does not span. For if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4$$

is any matrix in $\mathsf{span}(S_4)$, then $c = 2a$ since this dependency holds for each $\mathbf{v}_i$.

3.) Find a subset of S that is a basis for $W = \mathsf{span}(S)$:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -6 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \end{bmatrix} \right\}$$

Solution: To do this, we build a matrix with the given vectors $\mathbf{v}_j$ as its columns:

$$A = \begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 0 & 1 & 4 & 0 \\ -2 & 4 & 0 & -6 & 5 \\ 0 & 0 & -1 & -4 & 0 \end{bmatrix}.$$

Now we quickly row reduce this matrix to obtain

$$A \sim \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivots are in columns 1, 3 and 5, we take as our basis

$$S = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \end{bmatrix} \right\}.$$

4.) For each of the following, determine whether or not W is a subspace of the vector space P_3. If W **is** a subspace, prove that both properties hold. If W is **not** a subspace, show that some property fails using concrete examples.

(a) $W = \{at^3 + bt^2 + ct + d \ : \ ad = bc\}$

Solution: **NO** this is not a subspace. For example, both $p(t) = t + 1$ (with $a = b = 0, c = d = 1$) and $q(t) = t^3 + 2t^2 + 3t + 6$ belong to W, but

$$p(t) + q(t) = t^3 + 2t^2 + 4t + 7$$

does not since $1 \cdot 7 \neq 2 \cdot 4$. So W is not closed under addition.

(b) $W = \{at^3 + bt^2 + ct + d \ : \ a + d = b + c\}$

Solution: **YES** this is a subspace. First, suppose

$$p(t) = at^3 + bt^2 + ct + d \qquad \text{and} \qquad q(t) = a't^3 + b't^2 + c't + d'$$

belong to W. Then $a + d = b + c$ and $a' + d' = b' + c'$ giving us

$$(a + a') + (d + d') = (b + b') + (c + c')$$

which shows that

$$p(t) + q(t) = (a + a')t^3 + (b + b')t^2 + (c + c')t + (d + d')$$

belongs to W. Similarly, if r is any real number, then

$$r \cdot p(t) = (ra)t^3 + (rb)t^2 + (rc)t + (rd)$$

lies in W since $r^2ad = r^2bc$.

(c) $W = \{at^3 + bt^2 + ct + d \ : \ c \geq d\}$

Solution: **NO** this is not a subspace. for example, $p(t) = t$ lies in W since $c = 1$ and $d = 0$ here. But $(-1) \cdot p(t) = -t$ does not belong to W.

(d) $W = \{at^3 + bt^2 + ct + d \ : \ c = 0\}$

Solution: **YES** this is a subspace. First, suppose

$$p(t) = at^3 + bt^2 + d \qquad \text{and} \qquad q(t) = a't^3 + b't^2 + d'$$

belong to W. Then

$$p(t) + q(t) = (a + a')t^3 + (b + b')t^2 + (d + d')$$

also has vanishing constant term, so belongs to W. Similarly, if r is any real number, then

$$r \cdot p(t) = (ra)t^3 + (rb)t^2 + (rd)$$

lies in W.