C Term, Sections C01-C04 Instructor: W. J. Martin
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Matrices and Linear Algebra I – Test 2
Sample Solutions

1.) For each of the following functions L, determine whether or not L is a linear transformation. In all cases, give justification.

(a) [*3 points*] $L : \mathbb{R}^2 \to \mathbb{R}^3$ via $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2 \\ 0 \\ u_2 - 2 \end{bmatrix}$

Solution: This is NOT a linear transformation. One easy way to see it is to note that $L(\mathbf{0}) \neq \mathbf{0}$. Clearly,

$$L(\mathbf{0}) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

(b) [*3 points*] $L : \mathbb{R}^3 \to \mathbb{R}^2$ via $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 5u_2 - u_3 \\ u_3 - u_2 \end{bmatrix}$

Solution: Yes, this is a linear transformation. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Then $L(\mathbf{u} + \mathbf{v}) =$

$$L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} 5(u_2 + v_2) - (u_3 + v_3) \\ (u_3 + v_3) - (u_2 + v_2) \end{bmatrix} = \begin{bmatrix} (5u_2 - u_3) + (5v_2 - v_3) \\ (u_3 - u_2) + (v_3 - v_2) \end{bmatrix}$$

$= L(\mathbf{u}) + L(\mathbf{v})$ and similarly, for any scalar c,

$$L(c\mathbf{u}) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} 5(cu_2) - (cu_3) \\ (cu_3) - (cu_2) \end{bmatrix} = \begin{bmatrix} c(5u_2 - u_3) \\ c(u_3 - u_2) \end{bmatrix} = cL(\mathbf{u}).$$

(c) [*4 points*] $L : \mathbb{R} \to \mathbb{R}^2$ via $L\left(\begin{bmatrix} u \end{bmatrix}\right) = \begin{bmatrix} u \log u \\ u^3 - u \end{bmatrix}$

Solution: This is NOT a linear transformation. It is not even defined for $u < 0$ since you cannot take the logarithm of a negative number. To be clear, let $\mathbf{u} = [0]$. Then

$$L(\mathbf{0}) = \begin{bmatrix} 0 \cdot \log 0 \\ 0 \end{bmatrix}$$

which doesn't exist.

2.) In each case, determine whether or not the given vector $\mathbf{w}$ lies in the range of the given linear transformation L.

(a) *[5 points]* $L : \mathbb{R}^4 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} 4u_1 - u_2 - u_3 + 2u_4 \\ -2u_1 + \frac{1}{2}u_2 + \frac{1}{2}u_3 - u_4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Solution: We write down the augmented matrix $[A|\mathbf{w}]$ where A is the standard matrix representing L:

$$[A|\mathbf{w}] = \left[\begin{array}{cccc|c} 4 & -1 & -1 & 2 & 2 \\ -2 & \frac{1}{2} & \frac{1}{2} & -1 & 0 \end{array}\right]$$

and we row reduce to find a solution $\mathbf{u}$ satisfying $L(\mathbf{u}) = \mathbf{w}$.

$$[A|\mathbf{w}] \sim \left[\begin{array}{cccc|c} 4 & -1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R1 \\ R2 + \frac{1}{2}R1 \end{array}$$

We can stop here since it is now clear that the system admits no solution. We conclude "NO", $\mathbf{w}$ is not in the range of L.

(b) *[5 points]* $L : \mathbb{R}^3 \to \mathbb{R}^3$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 2u_1 - 6u_3 \\ u_2 - 3u_3 \\ -u_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$$

Solution: We write down the augmented matrix $[A|\mathbf{w}]$ where A is the standard matrix representing L:

$$[A|\mathbf{w}] = \left[\begin{array}{ccc|c} 2 & 0 & -6 & 8 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

and we row reduce to find a solution $\mathbf{u}$ satisfying $L(\mathbf{u}) = \mathbf{w}$.

$$[A|\mathbf{w}] \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \frac{1}{2}R1 \\ R2 \\ -R3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R1 + 3R3 \\ R2 + 3R3 \\ R3 \end{array}$$

This matrix is in r.r.e.f. So we can see that "YES" $\mathbf{w}$ is in the range of L. For if $\mathbf{u} = [1, 1, -1]^\top$, then $L(\mathbf{u}) = \mathbf{w}$.

3.) The following two questions pertain to planes in 3-dimensional space.

(a) [*6 points*] Find an equation describing the plane π passing through the points $(1, -1, -1)$, $(2, -1, -1)$ and $(0, 5, -4)$.

Solution: We write down the general equation

$$\pi : \; ax + by + cz = d$$

and solve a system of equations to find a, b, c, d. The equations come from the insistence that the plane passes through all three of these points:

$$\begin{aligned}(1)a + (-1)b + (-1)c - d &= 0\\ (2)a + (-1)b + (-1)c - d &= 0\\ (0)a + (5)b + (-4)c - d &= 0\end{aligned}$$

We obtain a homogeneous linear system with augmented matrix

$$[A|\mathbf{0}] = \left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0\\ 2 & -1 & -1 & -1 & 0\\ 0 & 5 & -4 & -1 & 0\end{array}\right]$$

After we row reduce this matrix, we obtain the matrix

$$[A|\mathbf{0}] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 1/3 & 0\\ 0 & 0 & 1 & 2/3 & 0\end{array}\right].$$

We want one simple non-trivial solution. So we take $d = -3$ and we find $a = 0$, $b = 1$, $c = 2$. An equation for our plane is then

$$\pi : y + 2z = -3.$$

(b) [*4 points*] Find an equation for the plane parallel to

$$\pi : 3x - 2y + z = -6$$

and passing through the point $(-2, 3, 12)$.

Solution: We know that parallel planes have the same normal vector. So the plane we seek has equation of the form

$$\sigma : 3x - 2y + z = d$$

for some d. To find d, we ensure that the plane passes through the given point: $3(-2) - 2(3) + 1(12) = 0$. So $d = 0$ and the answer is

$$3x - 2y + z = 0.$$

4.) [*10 points*] In each case, determine whether or not V with the given operations is a vector space. If it **fails** to be a vector space, show explicitly some part of the definition which fails. If you think it **is** a vector space, write down the zero vector.

(a) $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a, \ b \text{ any real numbers} \right\}$ with operations

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} b+d \\ a+c \end{bmatrix} \qquad \text{and} \qquad r \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra \\ rb \end{bmatrix}$$

Solution: This is **NOT** a vector space. For example, condition (f) fails: let $c = 2$, $d = 5$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$(c+d) \odot \mathbf{u} = 7 \odot \mathbf{u} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}.$$

But

$$(c \odot \mathbf{u}) \oplus (d \odot \mathbf{u}) = (2 \odot \mathbf{u}) \oplus (5 \odot \mathbf{u}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}.$$

So the condition

$$(c+d) \odot \mathbf{u} = (c \odot \mathbf{u}) \oplus (d \odot \mathbf{u})$$

fails in this case.

(b) $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a > 0, \ b > 0 \right\}$ with operations

$$\begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ bd \end{bmatrix} \qquad \text{and} \qquad r \odot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a^r \\ b^r \end{bmatrix}$$

Solution: This one is a vector space. (I can't find any part of the definition that fails.) The zero vector is

$$\mathbf{0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Indeed, for any vector $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, we have

$$\mathbf{0} \oplus \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1a \\ 1b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{u}$$

and $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$. (Note also that, since a and b are positive, the numbers $1/a$ and $1/b$ are positive. These have to be the entries of $-\mathbf{u}$. So $-\mathbf{u}$ is in V also.)