C Term, Sections C01-C04 Instructor: W. J. Martin
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Matrices and Linear Algebra I – Test 1

Sample Solutions

1.) [*10 points*] Consider the following system of linear equations:

$$\begin{array}{rrrrcr} 2x_1 & +4x_2 & -1x_3 & -3x_4 & = & 11 \\ x_1 & +2x_2 & +\ x_3 & & = & -2 \\ -x_1 & -2x_2 & +4x_3 & +5x_4 & = & -23 \end{array}$$

(a) Write down the augmented matrix corresponding to this system.

$$\left[\begin{array}{rrrr|r} 2 & 4 & -1 & -3 & 11 \\ 1 & 2 & 1 & 0 & -2 \\ -1 & -2 & 4 & 5 & -23 \end{array}\right]$$

(b) Perform Gauss-Jordan reduction on this matrix to obtain a matrix in **reduced row echelon form**.

$$\left[\begin{array}{rrrr|r} 2 & 4 & -1 & -3 & 11 \\ 1 & 2 & 1 & 0 & -2 \\ -1 & -2 & 4 & 5 & -23 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 0 & -2 \\ 0 & 0 & -3 & -3 & 15 \\ 0 & 0 & 5 & 5 & -25 \end{array}\right] \begin{array}{l} R2 \\ R1 - 2\cdot R2 \\ R3 + R2 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1 + \frac{1}{3}\cdot R2 \\ -\frac{1}{3}\cdot R2 \\ R3 + \frac{5}{3}\cdot R2 \end{array}$$

This final matrix is in r.r.e.f.

(c) Using part (b), find all solutions to the original linear system.

Here is a description of all possible solutions:

$$\begin{array}{rcl} x_1 & = & 3 - 2r + s, \\ x_2 & = & r, \text{ where } r \text{ is any real number} \\ x_3 & = & -5 - s, \\ x_4 & = & s, \text{ where } s \text{ is any real number} \end{array}$$

2.) [*10 points*] For each of the following matrices, either find its inverse or explain why it is not invertible.

(a) $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

Solution: We row reduce the partitioned matrix $[A|I]$:

$$[A|I] \sim \left[\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 4 & 9 & 2 & 1 & 0 \\ 0 & -4 & -9 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} R1 \\ R2 + 2 \cdot R1 \\ R3 - 3 \cdot R1 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{9}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R1 \\ \frac{1}{4} \cdot R2 \\ R3 + R2 \end{array}$$

We can stop here. Since there is a row of zeros to the left of the divider, we know that the **inverse does not exist**. Indeed, computing

$$\det(A) = 0 + 6 + (-16) - 0 - 2 - (-12) = 0$$

we see again that A is singular.

(b) $B = \begin{bmatrix} -1 & -6 & 6 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: We row reduce the partitioned matrix $[B|I]$:

$$[B|I] \sim \left[\begin{array}{rrr|rrr} 1 & 6 & -6 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -R1 \\ R2 \\ R3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R1 + 3 \cdot R2 \\ -\frac{1}{2} \cdot R2 \\ R3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R1 \\ R2 - R3 \\ -R3 \end{array}$$

So we find

$$B^{-1} = \begin{bmatrix} -1 & 3 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

3.) [*10 points*] Consider the matrix

$$B = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Clearly any matrix of the form rB commutes with B (where r is a real number) as do all matrices rI.

Find two matrices **not of this form** which commute with B.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want $AB = BA$. So we compute

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2a+b & a+2b \\ -2c+d & c+2d \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2a+c & -2b+d \\ a+2c & b+2d \end{bmatrix}.$$

We see that $b = c$ is forced. After that, there is only one equation really. It is

$$a + 2b = -2b + d.$$

Let's take $b = d = 1$ and $a = -3$ for our first matrix:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}.$$

For our second matrix, let's take $d = 0$ and $b = -1$ giving $a = 4$:

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}.$$

Thus we have two interesting examples of matrices A satisfying

$$AB = BA.$$

4.) Consider the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(a) *[5 points]* Show that, if $\mathbf{u}$ and $\mathbf{v}$ are solutions to the system, then $\mathbf{u} + \mathbf{v}$ is also a solution to the system.

Solution: Assume that $\mathbf{u}$ and $\mathbf{v}$ are solutions to the system $A\mathbf{x} = \mathbf{0}$. That is, suppose

$$A\mathbf{u} = \mathbf{0} \qquad \text{and} \qquad A\mathbf{v} = \mathbf{0}.$$

We compute

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= A\mathbf{u} + A\mathbf{v} \qquad \textit{(Distributive Law)} \\ &= \mathbf{0} + \mathbf{0} \qquad \textit{(since } \mathbf{u}, \mathbf{v} \textit{ are solutions)} \\ &= \mathbf{0} \end{aligned}$$

showing that $\mathbf{u} + \mathbf{v}$ is also a solution.

(b) *[5 points]* Show that, if $\mathbf{u}$ is a solution to the system and r is any real number, then $r\mathbf{u}$ is also a solution to the system.

Solution: Assume that $\mathbf{u}$ is a solution to the system $A\mathbf{x} = \mathbf{0}$ and that r is any real number. Then

$$A\mathbf{u} = \mathbf{0}.$$

We compute

$$\begin{aligned} A(r\mathbf{u}) &= r(A\mathbf{u}) \\ &= r\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

showing that $r\mathbf{u}$ is also a solution.