

## Matrices and Linear Algebra I – Test 1

### Sample Solutions

1.) [10 points] Consider the following system of linear equations:

$$\begin{array}{rrrrr} x_1 & -2x_2 & -5x_3 & -x_4 & = & 7 \\ -x_1 & & +x_3 & +x_4 & = & 3 \\ 2x_1 & +2x_2 & +2x_3 & & = & -4 \end{array}$$

(a) Write down the augmented matrix corresponding to this system.

$$\left[ \begin{array}{cccc|c} 1 & -2 & -5 & -1 & 7 \\ -1 & 0 & 1 & 1 & 3 \\ 2 & 2 & 2 & 0 & -4 \end{array} \right]$$

(b) Perform Gauss-Jordan reduction on this matrix to obtain a matrix in **reduced row echelon form**.

$$\begin{aligned} \left[ \begin{array}{cccc|c} 1 & -2 & -5 & -1 & 7 \\ -1 & 0 & 1 & 1 & 3 \\ 2 & 2 & 2 & 0 & -4 \end{array} \right] &\sim \left[ \begin{array}{cccc|c} 1 & -2 & -5 & -1 & 7 \\ 0 & -2 & -4 & 0 & 10 \\ 0 & 6 & 12 & 2 & -18 \end{array} \right] \begin{array}{l} R1 \\ R2 + R1 \\ R3 - 2 \cdot R1 \end{array} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & -1 & -1 & -3 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 2 & 12 \end{array} \right] \begin{array}{l} R1 - R2 \\ -\frac{1}{2} \cdot R2 \\ R3 + 3 \cdot R2 \end{array} \\ &\sim \left[ \begin{array}{cccc|c} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 6 \end{array} \right] \begin{array}{l} R1 + \frac{1}{2} \cdot R3 \\ R2 \\ \frac{1}{2} \cdot R3 \end{array} \end{aligned}$$

This final matrix is in r.r.e.f.

(c) Using part (b), find all solutions to the original linear system.

Here is a description of all possible solutions:

$$\begin{aligned} x_1 &= 3 + r, \\ x_2 &= -5 - 2r, \\ x_3 &= r, \text{ where } r \text{ is any real number} \\ x_4 &= 6. \end{aligned}$$

2.) [10 points] Consider the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Clearly any matrix of the form  $rB$  commutes with  $B$  (where  $r$  is a real number) as do all matrices  $rI$ .

Find two matrices **not of this form** which commute with  $B$ .

*Solution:* Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . We want  $AB = BA$ . So we compute

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2b & 2a-b \\ c+2d & 2c-d \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 2a-c & 2b-d \end{bmatrix}.$$

We see that  $b = c$  is forced. After that, there is only one equation really. It is

$$2a - b = b + 2d.$$

Let's take  $b = d = 1$  and  $a = 2$  for our first matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

For our second matrix, let's take  $d = 0$  and  $a = b = 5$ :

$$A = \begin{bmatrix} 5 & 5 \\ 5 & 0 \end{bmatrix}.$$

Thus we have two interesting examples of matrices  $A$  satisfying

$$AB = BA.$$

3.) Consider the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$ .

(a) [5 points] Show that, if  $\mathbf{u}$  and  $\mathbf{v}$  are solutions to the system, then  $\mathbf{u} + \mathbf{v}$  is also a solution to the system.

*Solution:* Assume that  $\mathbf{u}$  and  $\mathbf{v}$  are solutions to the system  $A\mathbf{x} = \mathbf{0}$ . That is, suppose

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}.$$

We compute

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= A\mathbf{u} + A\mathbf{v} && \text{(Distributive Law)} \\ &= \mathbf{0} + \mathbf{0} && \text{(since } \mathbf{u}, \mathbf{v} \text{ are solutions)} \\ &= \mathbf{0} \end{aligned}$$

showing that  $\mathbf{u} + \mathbf{v}$  is also a solution.

(b) [5 points] Show that, if  $\mathbf{u}$  is a solution to the system and  $r$  is any real number, then  $r\mathbf{u}$  is also a solution to the system.

*Solution:* Assume that  $\mathbf{u}$  is a solution to the system  $A\mathbf{x} = \mathbf{0}$  and that  $r$  is any real number. Then

$$A\mathbf{u} = \mathbf{0}.$$

We compute

$$\begin{aligned} A(r\mathbf{u}) &= r(A\mathbf{u}) \\ &= r\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

showing that  $r\mathbf{u}$  is also a solution.

4.) [10 points] For each of the following matrices, either find its inverse or explain why it is not invertible.

(a)  $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

*Solution:* We row reduce the partitioned matrix  $[A|I]$ :

$$\begin{aligned} [A|I] &\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} R1 \\ R2 - 2 \cdot R1 \\ R3 - 3 \cdot R1 \end{array} \\ &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{array} \right] \begin{array}{l} R1 + R3 \\ -R3 \\ R2 - 2 \cdot R3 \end{array} \\ &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{array} \right] \begin{array}{l} R1 + R3 \\ R2 \\ -R3 \end{array} \end{aligned}$$

So we find

$$A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -1 \\ -4 & -1 & 2 \end{bmatrix}.$$

(b)  $B = \begin{bmatrix} 2 & -2 & -12 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$

*Solution:* We row reduce the partitioned matrix  $[B|I]$ :

$$\begin{aligned} [B|I] &\sim \left[ \begin{array}{ccc|ccc} 1 & -1 & -6 & \frac{1}{2} & 0 & 0 \\ 0 & -1 & -6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \frac{1}{2}R1 \\ R2 \\ R3 \end{array} \\ &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 6 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R1 - R2 \\ -R2 \\ R3 \end{array} \\ &\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R1 \\ R2 - 6 \cdot R3 \\ R3 \end{array} \end{aligned}$$

So we find

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 0 & -1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$