Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Quiz 3

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ 2u_2 - u_3 \\ 2u_1 + u_3 \end{bmatrix}.$$

For each of the following vectors $\mathbf{v}$, determine whether or not $\mathbf{v}$ is in the *range* of L. If so, exhibit a vector $\mathbf{u} \in \mathbb{R}^3$ satisfying $L(\mathbf{u}) = \mathbf{v}$.

1.) $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution: YES. $\mathbf{v}$ is in the range since $\mathbf{v} = \mathbf{0}$ and any linear transformation satisfies $L(\mathbf{0}) = \mathbf{0}$.

2.) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Solution: YES. To decide whether there is a vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ satisfying $L(\mathbf{u}) = \mathbf{v}$, we solve the linear system

$$\begin{array}{rrrcl} u_1 & +u_2 & & = & 2 \\ & 2u_2 & -u_3 & = & 1 \\ 2u_1 & & +u_3 & = & 3 \end{array}$$

via Gauss-Jordan elimination. We have

$$\begin{aligned} [A|b] &= \left[\begin{array}{rrr|r} 1 & 1 & 0 & 2 \\ 0 & 2 & -1 & 1 \\ 2 & 0 & 1 & 3 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & 0 & 2 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 1 & -1 \end{array}\right] \begin{array}{l} R1 \\ R2 \\ R3 - 2 \cdot R1 \end{array} \\ &\sim \left[\begin{array}{rrr|r} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1 - \frac{1}{2} \cdot R2 \\ \frac{1}{2} \cdot R2 \\ R3 + R2 \end{array} \end{aligned}$$

This last matrix is in r.r.e.f. So a solution exists. For example, if we take $\mathbf{u} = [1, 1, 1]^\top$, then $L(\mathbf{u}) = \mathbf{v}$. (Why didn't I see that one right away?!)

3.) $\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Solution: NO. We have the same process as above, except that the values on the right-hand side are different:

$$\begin{aligned}
[A|b] &= \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 3 \end{array}\right] \begin{array}{l} R1 \\ R2 \\ R3 - 2 \cdot R1 \end{array} \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & -1 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right] \begin{array}{l} R1 - \frac{1}{2} \cdot R2 \\ \frac{1}{2} \cdot R2 \\ R3 + R2 \end{array}
\end{aligned}$$

This last matrix is not yet in r.r.e.f. But we can stop since it is now clear that no solution exists.

4.) $\mathbf{v} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$

Solution: Solution: YES. Again, we row reduce the augmented matrix where the RHS is $[0, 5, -5]^\top$:

$$\begin{aligned}
[A|b] &= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 5 \\ 2 & 0 & 1 & -5 \end{array}\right] \\
&\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 5 \\ 0 & -2 & 1 & -5 \end{array}\right] \begin{array}{l} R1 \\ R2 \\ R3 - 2 \cdot R1 \end{array} \\
&\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & -5/2 \\ 0 & 1 & -1/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1 - \frac{1}{2} \cdot R2 \\ \frac{1}{2} \cdot R2 \\ R3 + R2 \end{array}
\end{aligned}$$

This last matrix is in r.r.e.f. So a solution exists. For example, if we take $\mathbf{u} = [-3, 3, 1]^\top$, then $L(\mathbf{u}) = \mathbf{v}$.