Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 7

1.) Suppose A is a 4×4 matrix with eigenvalues $4, 2, 2, 1$.
(a) What are the eigenvalues of $8A$?

Solution: The eigenvalues of $8A$ are $32, 16, 16, 8$. To prove this, we note that for any $n \times n$ matrix M and any real number k, $\det(kM) = k^n \det(M)$. So $\det(8A - (8x)I) = 8^4 f_A(x)$ where f_A is the characteristic polynomial of A. It follows (for instance, by substituting $x = \frac{1}{8}\lambda$) that

$$\det(8A - \lambda I) = 8^4 \left[(\frac{1}{8}\lambda - 4)(\frac{1}{8}\lambda - 2)^2(\frac{1}{8}\lambda - 1) \right] = (\lambda - 32)(\lambda - 16)^2(\lambda - 8).$$

(b) What are the eigenvalues of A^3?

Solution: The eigenvalues of A^3 are **probably** $64, 8, 8, 1$. To prove this, let us assume that the eigenspace associated to $\lambda = 2$ is two-dimensional. Then we have a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ for $\mathbb{R}^4$ with $A\mathbf{v}_1 = 4\mathbf{v}_1$, $A\mathbf{v}_2 = 2\mathbf{v}_2$, $A\mathbf{v}_3 = 2\mathbf{v}_3$ and $A\mathbf{v}_4 = \mathbf{v}_4$. Then, for instance, we have

$$A^3\mathbf{v}_1 = A^2(A\mathbf{v}_1) = A^2(4\mathbf{v}_1) = 4A(A\mathbf{v}_1) = 4A(4\mathbf{v}_1) = 16(A\mathbf{v}_1) = 64\mathbf{v}_1.$$

Similar calculations give $A^3\mathbf{v}_2 = 8\mathbf{v}_2$, $A^3\mathbf{v}_3 = 8\mathbf{v}_3$ and $A^3\mathbf{v}_4 = \mathbf{v}_4$. Thus we have located four eigenvalues of A^3 and there can be no more. (GAP: What happens if that eigenspace has dimension only one?? I don't know right now!)

(c) What are the eigenvalues of A^T?

Solution: Theorem 3.1 tells us that $\det(B^\top) = \det(B)$ for any square matrix B. So if we apply this to $B = A - 4I$, $B = A - 2I$ and $B = A - I$, then we see that $A^\top$ has eigenvalues 4, 2 and 1 just as A does. Now the only challenge is to show that eigenvalue two still has multiplicity two. Well certainly there can be no other eigenvalues since $(A^\top)^\top = A$. Ah, but the row rank of $A^\top - 2I$ is the column rank of $A - 2I$ which we know is the same as the row rank of $A - 2I$ which is assumed to be two. So, yes, this eigenspace has dimension two as required.

2.) Exercise #21 on page 354 **except** use the following Leslie matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 6 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

Solution: We don't know the total population of this colony of this organism, but we can determine the relative ratios of the age groups in any stable population. That is, we seek a non-zero (all-positive) vector $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{x}$. Writing

$$\mathbf{x} = [x_0,\ x_1,\ x_2,\ x_3]^\top$$

we have

$$A\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This gives us a fairly simple linear system:

$$x_0 = 6x_3, \qquad x_1 = \frac{1}{3}x_0, \qquad x_2 = \frac{1}{\sqrt{2}}x_1, \qquad x_3 = \frac{1}{\sqrt{2}}x_2.$$

One solution is

$$\mathbf{x} = [6,\ 2,\ \sqrt{2},\ 1]^\top.$$

So the stable population is in the ratio 6 : 2 : 1.414 : 1. For example, if there are one million insects in the colony, then we expect roughly 576,136 newborns, 192,045 one-year-olds, 135,796 two-year-olds and 96,023 adult (age at least 3) females.

3.) Exercise #T.6 on page 355

Solution: We are given that $A_{n\times n}$ is nilpotent and we are asked to show that zero is the only eigenvalue of A. Indeed, suppose λ is any eigenvalue of A. Then there exists a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Suppose now that $A^k = O$ for some positive integer k. Now we compute

$$A^k\mathbf{x} = A^{k-1}(A\mathbf{x}) = \lambda A^{k-1}\mathbf{x} = \lambda^2 A^{k-2}\mathbf{x} = \cdots = \lambda^{k-1}A\mathbf{x} = \lambda^k\mathbf{x}.$$

But $A^k\mathbf{x} = O\mathbf{x} = \mathbf{0}$. So we have

$$\mathbf{0} = \lambda^k\mathbf{x}$$

with $\mathbf{x} \neq \mathbf{0}$, forcing $\lambda^k = 0$. This shows that every eigenvalue λ of A satisfies $\lambda = 0$.

4.) Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

Solution: We compute $f_A(\lambda) = \det(A - \lambda I)$:

$$f_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = -(\lambda - 5)(\lambda - 2)^2.$$

So the eigenvalues of A are 5, 2 and 2.

(b) Exhibit two <u>different</u> bases for $\mathbb{R}^3$ consisting solely of eigenvectors for A.

Solution: We need a basis for $\mathsf{nullspace}(A - \lambda I)$ for each eigenvalue λ. For $\lambda = 5$, we have

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

. So this eigenspace has dimension one and a basis is $S_5 = \{(1,1,1)^\top\}$.

For $\lambda = 2$, we have

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

. So this eigenspace has dimension two and a basis is $S_2 = \{(-1,1,0)^\top, (-1,0,1)^\top\}$.

Putting these together, we obtain our first basis for $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

There are many ways to transform this into a second basis. I'd like to choose an **orthonormal basis**, i.e., a basis of pairwise orthogonal unit vectors. To do this, I'll use the Gram-Schmidt procedure of Section 6.8. We obtain

$$T = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$