Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 6

1.) Exercise #14 on page 273.

Solution: We must find a basis for the space W spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ or

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

We may do this by determining all linear dependencies among the vectors. Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}.$$

That is,

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us a linear system

$$\begin{array}{ccccccccc} c_1 & & & + & c_3 & - & c_4 & = & 0 \\ & & c_2 & + & c_3 & + & c_4 & = & 0 \\ & & c_2 & + & c_3 & + & c_4 & = & 0 \\ c_1 & & & + & c_3 & - & c_4 & = & 0 \end{array}$$

with solutions

$$c_1 = s - r, \qquad c_2 = -r - s, \qquad c_3 = r, \qquad c_4 = s$$

where r and s can be any real numbers. This shows that

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

spans W since the last two matrices can be expressed as a linear combination of these. It is also clear that these are independent. So this is a basis for W and W has dimension two.

2.) Exercise #T.10 on page 274.

Solution: We are given that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for vector space V. So V has dimension 3. It is therefore enough to verify that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a linearly independent set. To do this, suppose c_1, c_2, c_3 are real numbers satisfying

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

in V. Then, substituting $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_3 = \mathbf{v}_1$, we find

$$(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + c_1\mathbf{v}_3 = \mathbf{0}.$$

But $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent! So we must have

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_1 &= 0 \end{aligned}$$

It is not hard now to see that $c_1 = c_2 = c_3 = 0$ is forced. This proves that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a linearly independent set in V. Since it has size $\mathsf{dim}(V)$, we may conclude it is a basis.

3.) We must devise a method to find a basis for

$$\mathsf{span}(S) \cap \mathsf{span}(T)$$

where

$$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \qquad \text{and} \qquad T = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$$

are linearly independent subsets in $\mathbb{R}^n$.

$$W = \mathsf{span}(S) \cap \mathsf{span}(T).$$

(Here, $\cap$ denotes the intersection of two sets, the set of all vectors that belong to both.)

HINT: The vectors in W are determined by the solutions to the equation

Solution: We first note that any vector $\mathbf{u}$ belongs to $\mathsf{span}(S)$ if it can be expressed as a linear combination of the vectors $\mathbf{v}_j$. Similarly, $\mathbf{u}$ lies in $\mathsf{span}(T)$ if $\mathbf{u}$ can be expressed as a linear combination of the $\mathbf{w}_j$. To get $\mathbf{u}$ in both spaces, we want $\mathbf{u}$ to be expressible in both ways. So we have

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + \cdots + d_\ell\mathbf{w}_\ell$$

for some scalars $c_1, \ldots, c_k$ and $d_1, \ldots, d_\ell$. We convert this to a homogeneous linear system:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k - d_1\mathbf{w}_1 - d_2\mathbf{w}_2 - \cdots - d_\ell\mathbf{w}_\ell = \mathbf{0}.$$

This is solved by the following algorithm (or method):

- Form the matrix A with columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$ followed by columns $-\mathbf{w}_1, \ldots, -\mathbf{w}_\ell$
- Row reduce the augmented matrix

$$[A\ |\mathbf{0}] = [\mathbf{v}_1 | \cdots | \mathbf{v}_k | -\mathbf{w}_1 | \cdots | -\mathbf{w}_\ell | \mathbf{0}]\,.$$

- Find a basis for the solution space to this homogeneous system (i.e., the null space of A)

- Each basis vector $x = [c_1, \ldots, c_k, d_1, \ldots, d_\ell]$ gives a vector $\mathbf{u}$ in $\mathsf{span}(S) \cap \mathsf{span}(T)$; specifically,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + \cdots + d_\ell\mathbf{w}_\ell.$$

- So a basis for $W = \mathsf{span}(S) \cap \mathsf{span}(T)$ is obtained as follows: for each basis vector for $\mathsf{nullspace}(A)$, take the last ℓ entries (call them $d'_1, d'_2, \ldots, d'_\ell$) and form the vector

$$\mathbf{u} = d'_1\mathbf{w}_1 + d'_2\mathbf{w}_2 + \cdots + d'_\ell\mathbf{w}_\ell$$

and place this vector $\mathbf{u}$ in the basis for space W. Thus the dimension of W is the dimension of $\mathsf{nullspace}(A)$.

4.) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -34 & -9 & 8 & 12 \\ -18 & -6 & 7 & 6 \\ -22 & -6 & 4 & 9 \end{bmatrix}.$$

We must find bases for the nullspaces of $A - I$, $A - 2I$ and $A - 3I$.

(a) Find a basis for $\mathsf{nullspace}(A - I)$.

Solution: We row reduce $[A - I|\mathbf{0}]$:

$$A = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ -34 & -10 & 8 & 12 & 0 \\ -18 & -6 & 6 & 6 & 0 \\ -22 & -6 & 4 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & -5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so a basis for this eigenspace is $S = \left\{ \begin{bmatrix} -1 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

(b) Find a basis for $\mathsf{nullspace}(A - 2I)$.

Solution: We row reduce $[A - 2I|\mathbf{0}]$:

$$A = \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ -34 & -11 & 8 & 12 & 0 \\ -18 & -6 & 5 & 6 & 0 \\ -22 & -6 & 4 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

which means that the linear system has only the trivial solution. The basis for the trivial subspace $\{\mathbf{0}\}$ is, by convention, the empty set.

(c) Find a basis for $\mathsf{nullspace}(A - 3I)$.

Solution: We row reduce $[A - 3I|\mathbf{0}]$:

$$A = \left[\begin{array}{rrrr|r} -2 & 0 & 0 & 0 & 0 \\ -34 & -12 & 8 & 12 & 0 \\ -18 & -6 & 4 & 6 & 0 \\ -22 & -6 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so a basis for this eigenspace is $S = \left\{ \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.