Linear Algebra
C Term, Sections C01-C04
W. J. Martin
February 18, 2003

Sample Solutions – Assignment 5

1.) Exercise #14 on page 250.

Solution:

(a) I claim that the set of all $n \times n$ symmetric matrices form a subspace of the vector space M_{nn}. First I note that this set is not empty since the 0 matrix is symmetric. Now I need only verify closure under addition and scalar multiplication.

Let A and B be symmetric $n \times n$ matrices. Then, by definition, $A^\top = A$ and $B^\top = B$. We use the appropriate theorem from Chapter 1:

$$(A + B)^\top = A^\top + B^\top = A + B$$

showing that $A + B$ is symmetric. So this set is closed under addition. Similarly, Chapter 1 tells us that $(cA)^\top = cA^\top$ for any scalar c. So if $A^\top = A$ and c is any real number, we have

$$(cA)^\top = cA^\top = cA$$

showing that cA is symmetric. This verifies closure under scalar multiplication.

(b) One way to see that the set of non-singular matrices is **not** a subspace of M_{nn} is to note that the zero matrix is singular. We know that any subspace of a vector space V must contain the zero vector of V. But in this case, that fails. So this cannot be a subspace.

(c) [**Sorry, I recalled the problem incorrectly:**] We will show that the set of all $n \times n$ skew symmetric matrices form a subspace of the vector space M_{nn}. First note that the zero matrix O is skew symmetric, so this set is nonempty.

Let A and B be skew symmetric $n \times n$ matrices. Then, by definition, $A^\top = -A$ and $B^\top = -B$. We again use the theorems from Chapter 1:

$$(A + B)^\top = A^\top + B^\top = -A + (-B) = -(A + B)$$

showing that $A + B$ is skew symmetric. So condition (α) holds. Similarly, if $A^\top = -A$ and c is any real number, we have

$$(cA)^\top = cA^\top = c(-A) = -(cA)$$

showing that cA is skew symmetric. This verifies condition (β).

2.) Exercise #T.8 on page 251.

Solution: Let A be an $m \times n$ matrix. The set W of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ such that $A_x \neq 0$ is **not** a subspace of $\mathbb{R}^n$. One quick way to see this is to note that $A\mathbf{0} = \mathbf{0}$ so that this set does not contain the zero vector. But any subspace of $\mathbb{R}^n$ must contain $\mathbf{0}$.

3.) Exercise #12 on page page 261.

Solution:

(a) We are given the vectors $p_1(t) = t^2 + 1$, $p_2(t) = t - 2$, $p_3(t) = t + 3$ in vector space P_2 and must decide whether or not they are dependent. We set up the equation

$$c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = \mathbf{0}$$

where $\mathbf{0} = 0t^2 + 0t + 0$ is the zero vector in P_2. Two polynomials are equal only when all corresponding coefficients are equal. So we obtain a system of three equations in three unknowns: and have

$$\begin{array}{ccccccc} c_1 & & & & & = & 0 \\ & & c_2 & + & c_3 & = & 0 \\ c_1 & - & 2c_2 & + & 3c_3 & = & 0 \end{array}$$

Solving this system by Gauss-Jordan elimination, we find it has unique solution $c_1 = c_2 = c_3 = 0$. Thus the given vectors are linearly independent.

(b) Similar to the previous part, we set up an equation of the form

$$c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = \mathbf{0}$$

and have

$$\begin{array}{ccccc} 2c_1 & + & c_2 & & = 0 \\ & & & c_3 & = 0 \\ c_1 & + & 3c_2 & & = 0 \end{array}$$

We get only one solution: $c_1 = c_2 = c_3 = 0$. Thus the given vectors are linearly independent.

(c) Again, we write down our generic equation and, using these vectors, we get the system

$$\begin{array}{ccccccc} & & 3c_2 & + & 2c_3 & = & 0 \\ 3c_1 & + & & & c_3 & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array}$$

By elimination, we obtain

$$\begin{array}{ccccccc} & & 3c_2 & + & 2c_3 & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0 \end{array}$$

a homogeneous system of two equations in three unknowns. So we know that a nontrivial solution exists. So the given vectors are linearly dependent. One non-trivial solution is

$$c_1 = 1, \ c_2 = 2, \ c_3 = -3$$

which means

$$p_1(t) + 2p_2(t) - 3p_3(t) = 0$$

So we can write

$$p_1(t) = -2p_2(t) + 3p_3(t)$$

(d) Using the same approach, we obtain the system

$$\begin{array}{rcrcrcl} c_1 & + & 5c_2 & + & 3c_3 & = & 0 \\ & - & 5c_2 & - & 5c_3 & = & 0 \\ -4c_1 & - & 6c_2 & + & 2c_3 & = & 0 \end{array}$$

By Gauss-Jordan elimination, we obtain

$$\begin{array}{rcrcrcl} c_1 & - & & + & 2c_3 & = & 0 \\ & & c_2 & + & c_3 & = & 0 \end{array}$$

So we know that a nontrivial solution exists. So the given vectors are linearly dependent. One non-trivial solution is

$$c_1 = 2,\ c_2 = -1,\ c_3 = 1$$

which means

$$2p_1(t) - p_2(t) + p_3(t) = 0$$

So,

$$p_2(t) = 2p_1(t) + p_3(t)$$

4.) Exercise #T.2 on page 262.

Proof: Throughout, assume

$$S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

and

$$S_2 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\}$$

for some $\ell \geq k$.

(a) Assume that S_1 is a linearly dependent set. Then there exist scalars $c_1, c_2, \ldots, c_k$ <u>not all zero</u> such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

in vector space V. Now define scalars $c'_1, c'_2, \ldots, c'_\ell$ by

$$c'_j = \begin{cases} c_j, & \text{if } 1 \leq j \leq k; \\ 0, & \text{if } k < j \leq \ell. \end{cases}$$

Then we have

$$\begin{array}{rcl} c'_1\mathbf{v}_1 + c'_2\mathbf{v}_2 + \cdots + c'_\ell\mathbf{v}_\ell & = & \\ c'_1\mathbf{v}_1 + \cdots + c'_k\mathbf{v}_k + c'_{k+1}\mathbf{v}_{k+1} + \cdots + c'_\ell\mathbf{v}_\ell & = & \\ c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_\ell & = & \\ c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k & = & \mathbf{0} \end{array}$$

Since some of the c_j' values are non-zero (namely at least one in the range $1 \le j \le k$, as given), this shows that S_2 is a linearly dependent set.

(b) *[This statement is the contrapositive of the previous one. So logically it is equivalent to part (a) and a separate proof is not needed. But we give one anyway.]*

Assume that S_2 is a linearly independent set. Now suppose $c_1, c_2, \ldots, c_k$ are any scalars such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Now define scalars $c_1', c_2', \ldots, c_\ell'$ by

$$c_j' = \begin{cases} c_j, & \text{if } 1 \le j \le k; \\ 0, & \text{if } k < j \le \ell. \end{cases}$$

Then we have

$$\begin{aligned} c_1'\mathbf{v}_1 + c_2'\mathbf{v}_2 + \cdots + c_\ell'\mathbf{v}_\ell &= \\ c_1'\mathbf{v}_1 + \cdots + c_k'\mathbf{v}_k + c_{k+1}'\mathbf{v}_{k+1} + \cdots + c_\ell'\mathbf{v}_\ell &= \\ c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_\ell &= \\ c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k &= \mathbf{0} \end{aligned}$$

But S_2 is independent! So all of the c_j' are equal to zero, including not only the ones we chose to be zero, but also the unknown constants $c_1, \ldots, c_k$. I.e., given any $c_1, \ldots, c_k$ which yield

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

we have shown that $c_1 = c_2 = \cdots = c_k = 0$ is forced. That is, we have shown that S_1 is a linearly independent set.