Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 4

1. (a) Find the equation of the plane in $\mathbb{R}^3$ which passes the points $(2,0,0)$, $(1,0,2)$ and $(-2,1,9)$.

   Solution: Suppose the plane π given by the equation

$$ax + by + cz = d$$

   passes through the points $(2,0,0)$, $(1,0,2)$ and $(-2,1,9)$. Then we have

$$\begin{array}{rcrcrcrcr} 2x & & & & & - & d & = & 0 \\ x & & & + & 2z & - & d & = & 0 \\ -2x & + & y & + & 9z & - & d & = & 0 \end{array}$$

   Solving this system by Gauss-Jordan elimination, we find

$$a = \frac{1}{2}r, \qquad b = -\frac{1}{4}, \qquad c = \frac{1}{4}r, \qquad d = r \qquad \text{where } r \text{ is any real number.}$$

   Letting $r = 4$, we obtain $a = 2,\ b = -1,\ c = 1,\ d = 4$. Thus, an equation for the desired plane is

$$2x - y + z = 4.$$

   (b) Find the equation of the plane which is parallel to the plane $\pi : 3x + y - 4z = 7$ and passes through the point $(1,1,1)$.

   Solution: Parallel planes have parallel normal vectors. So we know that for the equation of any plane parallel to $\pi :\ 3x + y - 4z = 7$ has the form

$$3x + y - 4z = d$$

   for some number d. Now the point $(1,1,1)$ lies on our new plane only if it satisfies the defining equation. So we plug in the coordinates of our point to find d:

$$3(1) + (1) - 4(1) = d,$$

   giving $d = 0$. So one possible equation for this plane is

$$3x + y - 4z = 0.$$

(c) Find the equation of the plane which is perpendicular to the vector $(2, -1, 1)$ and passes through the point $(0, -3, 1)$.

Solution: Similar to the previous part, we already have the parameters a, b, c; our plane has an equation of the form

$$2x - y + z = d.$$

In order to find d, we plug in the given point $(0, -3, 1)$:

$$d = 2(0) - (-3) + (1) = 4.$$

We obtain the following equation for our little plane:

$$2x - y + z = 4.$$

2. Exercise #16 on page 243.

Solution: Our $\oplus$ is the ordinary one, so we know that (α) and (a)–(d) will hold (since $\mathbb{R}^2$ is a vector space!). But $\odot$ seems weird. In fact, condition (h) of the definition fails. To wit, let $\mathbf{u} = (3, 5)$. Then $1 \odot \mathbf{u} = (0, 0) \neq \mathbf{u}$. So this is **not** a vector space.

3. Exercise #18 on page page 244.

Solution: Since the scalar multiplication is the natural one, we focus our scrutiny on the strange addition given. This is not even commutative! Indeed, if $\mathbf{u} = 7$ and $\mathbf{v} = 3$, then

$$\mathbf{u} \oplus \mathbf{v} = 2 \cdot 7 - 3 = 11 \neq -1 = 2 \cdot 3 - 7 = \mathbf{v} \oplus \mathbf{u}.$$

So this is **not** a vector space. [Note that conditions (b), (c) and (f) also fail. Since there is no zero, it makes no sense to even consider condition (d).]

4. Exercise #T.3 on page 244.

Proof: Assume $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$. Then, adding $-\mathbf{u}$ to both sides, we get

$$\begin{aligned} -\mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{v}) &= -\mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{w}) \quad \text{(addition is well-defined)} \\ (-\mathbf{u} \oplus \mathbf{u}) \oplus \mathbf{v} &= (-\mathbf{u} \oplus \mathbf{u}) \oplus \mathbf{w} \quad \text{(using (b))} \\ \mathbf{0} \oplus \mathbf{v} &= \mathbf{0} \oplus \mathbf{w} \quad \text{(using (d))} \\ \mathbf{v} &= \mathbf{w} \quad \text{(using (c))} \end{aligned}$$