Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 3

1. We must decide, in parts (a)–(d), if L is a linear transformation.

Solution:
(a) $L : \mathbb{R}^2 \to \mathbb{R}^3$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \\ u_1u_2 \end{bmatrix}$$

This is **not** a linear transformation. Indeed, let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $L(\mathbf{u}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Now let $r = 2$. Then

$$r\,\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and we have

$$L(r\mathbf{u}) = L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = rL(\mathbf{u}).$$

So property (a) of the definition fails to hold.

(b) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_1 - u_2 \end{bmatrix}.$$

To show that L is a linear transformation, let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \ \textit{and}\ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then
(1)

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 + v_1 + v_2 \\ u_1 - u_2 + v_1 - v_2 \end{bmatrix}$$

On the other hand

$$L(\mathbf{u})+L(\mathbf{v}) = L\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}+L\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+u_2 \\ u_1-u_2 \end{bmatrix}+\begin{bmatrix} v_1+v_2 \\ v_1-v_2 \end{bmatrix} = \begin{bmatrix} u_1+u_2+v_1+v_2 \\ u_1-u_2+v_1-v_2 \end{bmatrix}$$

So $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$.

(2)

$$L(k\mathbf{u}) = L\left(k\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}\right) = \begin{bmatrix} ku_1+ku_2 \\ ku_1-ku_2 \end{bmatrix}$$

On the other hand

$$kL(\mathbf{u}) = k\left(L\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = k\begin{bmatrix} u_1+u_2 \\ u_1-u_2 \end{bmatrix} = \begin{bmatrix} ku_1+ku_2 \\ ku_1-ku_2 \end{bmatrix}$$

So $L(k\mathbf{u}) = kL(\mathbf{u})$.
From (1) and (2), we can conclude that L is linear.

(c) We have $L : \mathbb{R}^4 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} 2u_4 \\ 0 \end{bmatrix}.$$

This is a linear transformation. For if we let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \; and \; \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

Then
(1)

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+u_3 \\ u_4+u_4 \end{bmatrix}\right) = \begin{bmatrix} 2(u_4+v_4) \\ 0 \end{bmatrix}$$

On the other hand

$$L(\mathbf{u})+L(\mathbf{v}) = L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}+L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 2u_4 \\ 0 \end{bmatrix}+\begin{bmatrix} 2v_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(u_4+v_4) \\ 0 \end{bmatrix}$$

So $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$.

(2)

$$L(k\mathbf{u}) = L\left(k\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = L\left(\begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \\ ku_4 \end{bmatrix}\right) = \begin{bmatrix} 2ku_4 \\ 0 \end{bmatrix}$$

On the other hand

$$kL(\mathbf{u}) = k\left(L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = k\begin{bmatrix} 2u_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2ku_4 \\ 0 \end{bmatrix}$$

So $L(k\mathbf{u}) = kL(\mathbf{u})$.
From (1) and (2), we can conclude that L is linear.

(d) We are given $L : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 2u_1 + 5 \\ u_1 + u_2 \end{bmatrix}.$$

This is **not** a linear transformation. For we immediately see that $L(\mathbf{0}) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \neq \mathbf{0}$. So, by Corollary 4.1, L cannot be linear.

2. For each of the following linear transformations, write down the standard matrix representing L.

   (a) $L : \mathbb{R}^2 \to \mathbb{R}^3$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 2u_2 \\ 5u_1 - u_2 \end{bmatrix}$$

   (b) $L : \mathbb{R}^2 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 2u_2 \\ 2u_1 \end{bmatrix}$$

   (c) $L : \mathbb{R}^4 \to \mathbb{R}^2$ via

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 7u_3 - 2u_4 \\ 0 \end{bmatrix}$$

Solution:

(a) The standard matrix A representing L is the 3×2 matrix whose columns are $L(\mathbf{e_1})$, $L(\mathbf{e_2})$ respectively. Thus:

$$L(\mathbf{e_1}) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-0 \\ 0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = col_1(A)$$

$$L(\mathbf{e_2}) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = col_2(A)$$

Hence $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & -1 \end{bmatrix}$

(b) The standard matrix A representing L is the 2×2 matrix whose columns are $L(\mathbf{e_1})$, $L(\mathbf{e_2})$ respectively. Thus:

$$L(\mathbf{e_1}) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} == col_1(A)$$

$$L(\mathbf{e_2}) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = col_2(A)$$

Hence $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(c) The standard matrix A representing L is the 2×2 matrix whose columns are $L(\mathbf{e_1})$, $L(\mathbf{e_2})$ and $L(\mathbf{e_3})$ respectively. Thus:

$$L(\mathbf{e_1}) = L\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0-0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = col_1(A)$$

$$L(\mathbf{e_2}) = L\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0-0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = col_2(A)$$

$$L(\mathbf{e_3}) = L\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+7-0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix} = col_3(A)$$

$$L(\mathbf{e_4}) = L\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0-2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = col_4(A)$$

Hence $A = \begin{bmatrix} 1 & 0 & 7 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. Exercise #T.5 on pages 213.
   Solution: Because $L : R' \to R'$ be defined $L(\mathbf{u}) = a\mathbf{u} + b$, for each $\mathbf{u}$ or $\mathbf{v}$ in R^n, if L is a linear transformation, then according to the definition:
   (**1**) $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$;
   (**2**) $L(k\mathbf{v}) = kL(\mathbf{u})$;

   So we get
   (**1**) $L(\mathbf{u}+\mathbf{v}) = a(\mathbf{u}+\mathbf{v})+b = a\mathbf{u}+a\mathbf{v}+b = L\mathbf{u}+L\mathbf{v} = (a\mathbf{u}+b)+(a\mathbf{v}+b) = a\mathbf{u}+\mathbf{v}+2b$
   So $b = 0$
   (**2**) $L(k\mathbf{u}) = a(\mathbf{u}) + b = ak\mathbf{u} + b = kL\mathbf{u} = k(a\mathbf{u} + b) = ak\mathbf{u} + kb$
   So $b = kb$
   Exactly when $b = 0$, we have $b = kb$ for any k. So we can conclude that L is a linear transformation only when $b = 0$, but for any real number a.

4. Exercise #T.8 on page 214.
   Proof: Let $L : R^n \to R^n$ be a linear transformation and $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R^n}$, then according to Theorem 4.6:

   $$L(a\mathbf{u} + b\mathbf{v}) = a(L\mathbf{u}) + b(L\mathbf{v}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$$

   for any scalars a and b.