Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 2

Problem 1(a): The DCW company produces three models with resources and profits summarized as follows:

Model:	P_1	P_2	P_3	Total Resource Avail.
Steel per unit	2	2	1	1800
Plastic per unit	3	1	1	1500
Profit per unit	\$150	\$110	\$60	

We must first determine all possible production schedules which fully utilize the two resources.

Solution: Let us say that x_1 units of P_1 are to be produced, x_2 units of P_2 and x_3 units of P_3. The conditions that the resources are fully utilized lead to the following system of equations:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 1800 \\ 3x_1 + x_2 + x_3 &= 1500 \end{aligned}$$

We apply the method of elimination: subtracting the first equation from the second to obtain $x_1 - x_2 = -300$; subtracting 2 times the second equation from the first to obtain $-4x_1 - x_3 = -1200$. We transform those equations and get:

$$\begin{aligned} x_1 &= r \\ x_2 &= r + 300 \\ x_3 &= 1200 - 4r \end{aligned}$$

where r can be any real number. But the application forces us to consider only solutions $[x_1, x_2, x_3]$ with all entries greater than or equal to zero. So we find

$$0 \le r \le 300.$$

In summary, any x_1, x_2, x_3 which satisfy the problem's requirements are described by these equations with r in the given range.

(**b**) Among the solutions found in part (**a**), which production schedule will maximize profit?

Solution: Let w denote the profit we can obtain from a given production schedule $[x_1, x_2, x_3]$. Then

$$w = 150x_1 + 110x_2 + 60x_3,$$

which we can write entirely in terms of our free parameter r:

$$w = 105,000 + 20r.$$

This is our profit, in dollars. So we clearly want to make r as large as possible. Keeping the real-world problem in mind, we choose $r = 300$. Here is the optimum production schedule:

Model	P_1	P_2	P_3	**Total**
Units produced	300	600	0	900
Steel used	600	1200	0	1800
Plastic used	900	600	0	1500
Profit	\$45,000	\$66,000	\$0	\$111,000

Problem 2(a): Find all values of r for which the matrix $A = \begin{bmatrix} r & 3 & 3 \\ 3 & r & 3 \\ 3 & 3 & r \end{bmatrix}$ is singular.

Solution: We compute

$$\det(A) = r^3 + 27 + 27 - 9r - 9r - 9r = r^3 - 27r + 54 = (r+6)(r-3)^2.$$

So A is singular (non-invertible) if and only if its determinant is zero, i.e., if and only if $r = -6$ or $r = 3$.

(b) For all other values of r, compute A^{-1}.

Solution: **(a)** To find all value of r for which the matrix $\begin{bmatrix} r & 3 & 3 \\ 3 & r & 3 \\ 3 & 3 & r \end{bmatrix}$ is singular, we need to do the following basic operations on A:

Adding the second and third rows to the first row to obtain: $\begin{bmatrix} r+6 & r+6 & r+6 \\ 3 & r & 3 \\ 3 & 3 & r \end{bmatrix}$ Multiply the first row by $\frac{1}{r+6}$ to get: $\begin{bmatrix} 1 & 1 & 1 \\ 3 & r & 3 \\ 3 & 3 & r \end{bmatrix}$. Adding (-3) times the first row to the second row and third row to obtain: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & r-3 & 0 \\ 0 & 0 & r-3 \end{bmatrix}$. Multiply the second row and the third row by $\frac{1}{r-3}$ to get: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Finally we obtain I_3. So when $r \neq 3, r \neq -6$, the matrix A is row equivalent to I_3, which denotes A is nonsingular. When $r = 3 or r = -6$, matrix A is singular.

(b) Step1. We row reduce the 3×6 matrix

$$[A|I_3] = \begin{bmatrix} r & 3 & 3 & 1 & 0 & 0 \\ 3 & r & 3 & 0 & 1 & 0 \\ 3 & 3 & r & 0 & 0 & 1 \end{bmatrix}.$$

To avoid a mess, we will supress all divisions until the end of the row reduction algorithm. We get Adding the second and third rows to the first row to obtain:

$$[A|I] \sim \begin{bmatrix} r & 3 & 3 & 1 & 0 & 0 \\ 0 & r^2-9 & 3(r-3) & -3 & r & 0 \\ 0 & 3(r-3) & r^2-9 & -3 & 0 & r \end{bmatrix} \begin{matrix} \\ r \cdot R2 - 3 \cdot R1 \\ r \cdot R3 - 3 \cdot R1 \end{matrix}$$

Multiply the first row by $\frac{1}{r+6}$ to get:

$$\begin{bmatrix} 1 & 1 & 1 & \frac{1}{r+6} & \frac{1}{r+6} & \frac{1}{r+6} \\ 3 & r & 3 & 0 & 1 & 0 \\ 3 & 3 & r & 0 & 0 & 1 \end{bmatrix}.$$

Adding (-3) times the first row to the second row and third row to obtain:

$$\begin{bmatrix} 1 & 1 & 1 & \frac{1}{r+6} & \frac{1}{r+6} & \frac{1}{r+6} \\ 0 & r-3 & 0 & \frac{-3}{r+6} & 1 + \frac{-3}{r+6} & \frac{-3}{r+6} \\ 0 & 0 & r-3 & \frac{-3}{r+6} & \frac{-3}{r+6} & 1 + \frac{-3}{r+6} \end{bmatrix}.$$

Multiply the second row and the third row by $\frac{1}{r-3}$ to get:

$$\begin{bmatrix} 1 & 1 & 1 & \frac{1}{r+6} & \frac{1}{r+6} & \frac{1}{r+6} \\ 0 & 1 & 0 & \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3} + \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} \\ 0 & 0 & 1 & \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3} + \frac{-3}{(r+6)(r-3)} \end{bmatrix}.$$

Finally we obtain

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{r+6}+\frac{6}{(r+6)(r-3)} & \frac{1}{r+6}+\frac{6}{(r+6)(r-3)}-\frac{1}{r-3} & \frac{1}{r+6}+\frac{6}{(r+6)(r-3)}-\frac{1}{r-3} \\ 0 & 1 & 0 & \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3}+\frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} \\ 0 & 0 & 1 & \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3}+\frac{-3}{(r+6)(r-3)} \end{array}\right].$$

Step3: since $C = I_3$, we conclude $D = A^{-1}$. Hence

$$A^{-1} = \left[\begin{array}{ccc} \frac{1}{r+6}+\frac{6}{(r+6)(r-3)} & \frac{1}{r+6}+\frac{6}{(r+6)(r-3)}-\frac{1}{r-3} & \frac{1}{r+6}+\frac{6}{(r+6)(r-3)}-\frac{1}{r-3} \\ \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3}+\frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} \\ \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} & \frac{1}{r-3}+\frac{-3}{(r+6)(r-3)} \end{array}\right] = \left[\begin{array}{ccc} \frac{r+3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} \\ \frac{-3}{(r+6)(r-3)} & \frac{r+3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} \\ \frac{-3}{(r+6)(r-3)} & \frac{-3}{(r+6)(r-3)} & \frac{r+3}{(r+6)(r-3)} \end{array}\right]$$

Problem T.27 on page 89. We find square roots of three matrices and show that this is impossible for a fourth.

Solution:

(a) We are given

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and we seek a 2×2 matrix A which satisfies $A^2 = B$. Let's write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then our requirement is

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d \end{bmatrix} = B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

from which we obtain a nonlinear system:

$$\begin{aligned} a^2 + bc &= 1 \\ b(a + d) &= 1 \\ c(a + d) &= 0 \\ bc + d^2 &= 1 \end{aligned}$$

Now it is useful to remember that we need only find ONE square root. (We were not asked to find all solutions.) So we try $c = 0$ and our equations reduce to

$$a^2 = 1, \qquad b(a + d) = 1, \qquad d^2 = 1.$$

So $a = d = \pm 1$ and $b = \pm\frac{1}{2}$. So here is a square root of the given matrix B:

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

(b) We are given

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We try to make the problem simpler by first seeing if there is a matrix A with plenty of zero entries which satisfies $A^2 = B$. And indeed, our first attempt is successful:

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{yields} \qquad A^2 = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So we find two square roots of B having this special form, namely

$$A = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(c) Clearly $I^2 = I$, so I itself is a square root of the identity matrix I. But here is another one:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

(This is a type of reflection. There are many other "mirrors" about which we can reflect in 4-space. Here's another square root of I_4:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Of course, most square roots of I are not nearly so pretty.)

(d) In this part, we have to be more rigorous. We can't just say that we cannot find a square root of the given matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We must show that it is impossible for ANYONE to find it.

So, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then we need

$$A^2 = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We arrive at system of equations:

$$\begin{aligned} a^2 + bc &= 0 \\ b(a+d) &= 1 \\ c(a+d) &= 0 \\ bc + d^2 &= 0 \end{aligned}$$

Looking at the second equation, we see that $a + d \neq 0$. So the third equation tells us that $c = 0$. Then we are in trouble because the first equation now forces $a = 0$ and the last forces $d = 0$. But we already said that $a + d = 0$ is not healthy for the second equation! That is, we have a contradiction: the matrix A does not exist.

Problem T.12 on page 65. If $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then their difference $\mathbf{u} - \mathbf{v}$ is a solution to the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Solution: Assume $\mathbf{u}$ and $\mathbf{v}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$. That means that

$$A\mathbf{u} = \mathbf{b} \qquad \text{and} \qquad A\mathbf{v} = \mathbf{b}.$$

We now compute

$$\begin{aligned} A(\mathbf{u} - \mathbf{v}) &= A\mathbf{u} - A\mathbf{v} && \textit{(Distributive Law)} \\ &= \mathbf{b} - \mathbf{b} && \textit{(since they are solutions)} \\ &= \mathbf{0} \end{aligned}$$

showing that $\mathbf{u} - \mathbf{v}$ is indeed a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.