Linear Algebra
C Term, Sections C01-C04
W. J. Martin
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Sample Solutions – Assignment 1

Problem 1: Systematically determine all 2x2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which satisfy $A^2 = 0$.

Solution: Given: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$A^2 = AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}.$$

To satisfy $A^2 = 0$, the following equations should be met:

$$\begin{aligned} a^2 + bc &= 0 \\ b(a + d) &= 0 \\ c(a + d) &= 0 \\ cb + d^2 &= 0 \end{aligned}$$

We split the analysis into two cases.

Case I: Suppose $b = 0$. Then our second equation becomes vacuous and the remaining conditions are expressed as

$$a^2 = 0, \qquad c(a + d) = 0, \qquad d^2 = 0.$$

So we clearly have $a = d = 0$ and now c can be any real number. So our matrix takes the form

$$A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, \qquad c \in \mathbb{R}.$$

Case II: Suppose $b \neq 0$. (Clearly this is the only other possibility.) Then the second equation gives us $a + d = 0$. So we can replace d by $-a$. Our second and third equation are now trivial and the first and last are equivalent to

$$a^2 + bc = 0$$

which, given any a and b (with $b \neq 0$), is satified by taking $c = -a^2/b$. So our matrix takes the form

$$A = \begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix}, \qquad\qquad a, b \in \mathbb{R},\ b \neq 0.$$

Problem 2: A manufacturer makes 2-minute, 6-minute, and 9-minute film developers. Each ton of 2-minute developer requires 6 minutes in plant A and 24 minutes in plant B. Each ton of 6-minute developer requires 12 minutes in plant A and 12 minutes in plant B. Each ton of 9-minute developer requires 12 minutes in plant A and 12 minutes in plant B. If plant A is available 10 hours per day and plant B is available 16 hours per day, how many tons of each type of developer can be produced so that the plants are fully utilized?

Solution: Given that x tons of 2-minute developer, y tons of 6-minute developer, and z tons of 9-minute developer can be produced so that the plants are fully utilized, then from the statement of the problem, we obtain the following system of equations:

$$\begin{aligned} 6x + 12y + 12z &= 10 \cdot 60 = 600 \\ 24x + 12y + 12z &= 16 \cdot 60 = 960 \end{aligned}$$

Subtracting the second equation from the first, we obtain:

$$18x = 360.$$

So $x = 20$. Substituting $x = 20$ to the original equations, we obtain two identical equations:

$$\begin{aligned} 12y + 12z &= 480 \\ 12y + 12z &= 480 \end{aligned}$$

Multiply by 1/12 on both sides this equation to get

$$y + z = 40$$

Because y and z denote the numbers of tons, they should be nonnegative. So the original equation system is transformed into:

$$\begin{aligned} x &= 20 \\ y + z &= 40 \\ y,\ z &\geq 0 \end{aligned}$$

For any real numbers x, y, and z, which satisfy these simple conditions, there is a feasible production schedule to produce x tons of 2-minute developer, y tons of 6-minute developer, and z tons of 9-minute developer.

Problem 3: Matrices A and B are said to commute if AB=BA. Find all values a, b, c, d, e, f for which the $3x3$ matrix

$$A = \begin{bmatrix} a & 3 & b \\ 3 & c & d \\ e & 3 & f \end{bmatrix} \qquad \text{commutes with the matrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 9 \end{bmatrix}.$$

Solution: We compute both AB and BA and set the two matrices equal to one another:

$$AB = \begin{bmatrix} a+6+3b & 2a & 3a+9b \\ 3+2c+3d & 6 & 9+9d \\ e+6+3f & 2e & 3e+9f \end{bmatrix} \text{ while } BA = \begin{bmatrix} a+6+3e & 12+2c & b+2d+3f \\ 2a & 6 & 2b \\ 3a+9e & 36 & 3b+9f \end{bmatrix}.$$

The equation $AB = BA$ gives us a system of linear equations. Before writing the full system out, we make the obvious deductions $e = 18$ and $a = c+6$ which come from looking at the second column of each matrix. Now we have

$$\begin{aligned} c+12+3b &= c+66 \\ 9b+3c+18 &= b+2d+3f \\ 3+2c+3d &= 12+2c \\ 9+9d &= 2b \\ 24+3f &= 3c+180 \\ 54+9f &= 3b+9f \end{aligned}$$

Now we can quickly eliminate a few more unknowns: from the last equation, we see that $b = 54/3 = 18$ (this also satisfies the first equation) and the third equation gives us $d = 3$. We are now left with two unknowns — c and f — and these must satisfy $3c + 180 = 3f + 24$ (from row 1, column 3). So there are infinitely many matrices A which commute with B. For any real number c, we obtain a matrix

$$A = \begin{bmatrix} c+6 & 3 & 18 \\ 3 & c & 3 \\ 18 & 3 & c+52 \end{bmatrix}$$

and it is straigtforward to check that any matrix of this form indeed does commute with the given matrix B.

Problem 4: In this problem, we compute a few "dot products" on the space of all continuous functions on $[-\pi, \pi]$ where our inner product is:

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

If the value of this integral is zero, we say f and g are *"orthogonal"*. We are given the following four examples:

(**a**) $a(x) = \sin x$ for $-\pi \le x \le \pi$
(**b**) $b(x) = \cos x$ for $-\pi \le x \le \pi$
(**c**) $c(x) = x$ for $-\pi \le x \le \pi$
(**d**) $d(x) = 1$ for $-\pi \le x \le \pi$

Solution:

(**(a) and (b)**) We have

$$\langle a, b\rangle := \int_{-\pi}^{\pi} \sin x \cos x \, dx = \frac{1}{2}\int \sin(2x)\, dx\bigg|_{-\pi}^{\pi} = -\frac{1}{4}\cos(2x)\bigg|_{-\pi}^{\pi} = 0$$

so $a(x)$ and $b(x)$ are ORTHOGONAL.

(**(a) and (c)**) We have

$$\langle a, c\rangle := \int_{-\pi}^{\pi} x \sin x \, dx = \int x \sin x \, dx\bigg|_{-\pi}^{\pi} = \sin x - x\cos x\,|_{-\pi}^{\pi} = 2\pi$$

so $a(x)$ and $c(x)$ are NOT orthogonal.

(**(a) and (d)**) We have

$$\langle a, d\rangle := \int_{-\pi}^{\pi} \sin x \, dx = 0$$

so $a(x)$ and $d(x)$ are ORTHOGONAL.

(**(b) and (c)**) We have

$$\langle b, c\rangle := \int_{-\pi}^{\pi} x \cos x \, dx = \int x \cos x \, dx\bigg|_{-\pi}^{\pi} = \cos x + x\sin x\,|_{-\pi}^{\pi} = 0$$

so $b(x)$ and $c(x)$ are ORTHOGONAL.

(**(b) and (d)**) We have

$$\langle b, d\rangle := \int_{-\pi}^{\pi} \cos x \, dx = 0$$

so $b(x)$ and $d(x)$ are ORTHOGONAL.

(**(c) and (d)**) We have

$$\langle c, d\rangle := \int_{-\pi}^{\pi} x \, dx = 0$$

so $c(x)$ and $d(x)$ are ORTHOGONAL.